Lecture 14 & 15 — February 27 & March 4, 2013

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1 Graph Laplacians

Let $G = (V, E)$ be a graph with non-negative weights $w : E \to \mathbb{R}$. Let $e_i \in \mathbb{R}^n$ be the i^{th} standard basis vector, and e be the all ones vector.

Definition 1. Let $y_{uv} := e_u - e_v$, $y_{uv} := y_{uv} y_{uv}^T$. The Laplacian matrix of G is the matrix

$$L_G = \sum_{u,v \in E} w_{wv} \cdot Y_{uv}.$$

For the purposes of this document, we will need the following facts about Laplacians. For details, refer to the previous year's notes on Graph Laplacians.

Fact 2. *Let $G = (V, E)$ be a graph with non-negative weights $w : E \to \mathbb{R}$. Then the weighted Laplacian L_G is positive semi-definite.*

Fact 3. *If G is connected then $im(L_G) = \{x \,|\, \sum_i x_i = 0\}$ which is a $(n-1)$ dimensional subspace. In general, the dimension of the null space of the L_G is the number of connected components.*

2 Linear Algebra

Here we elaborate on last year's "Notes on Symmetric Matrices". Let $A^\dagger$ denotes pseudo inverse of A. If U and V are subspaces of $\mathbb{R}^n$, then we will let $U + V$ denote the span of $U \cup V$, (i.e., all linear combinations of vectors in U and V), or equivalently their Minkowski sum.

$$U + V \;:=\; \text{span}\,(U \cup V) = \{u + v \,|\, u \in U, v \in V\}$$

Finally, if G is a (weighted) graph, L_G denotes the graph Laplacian.

Fact 4. *Let $A, B, C \in \mathbb{R}^{n \times n}$ be symmetric matrices. Then*

$$B \succeq A \Rightarrow CBC \succeq CAC$$

If C is full rank, then the converse holds too.

Proof. This is a straightforward consequence of Fact 13. □

Fact 5. *Let $A, B_1, \ldots, B_k$ be symmetric PSD matrices. Suppose for all i $im(B_i) \subseteq im(A)$. Then for $l, u \in \mathbb{R}$*

$$l \cdot A \;\preceq\; \sum_i B_i \;\preceq\; u \cdot A \qquad \Longleftrightarrow \qquad l \cdot I_{im(A)} \;\preceq\; \sum_i A^{\dagger/2} B_i A^{\dagger/2} \;\preceq\; u \cdot I_{im(A)}$$

Proof. Since A is symmetric, $A^{\dagger/2}$ is also symmetric. Note that

$$\begin{aligned} \text{im}\left(\sum_i B_i\right) &\subseteq \text{span}\left(\bigcup_i \text{im} B_i\right) \subseteq \text{im}(A) = \text{im}(A^{\dagger/2}) \\ \Longrightarrow \quad & \text{im}(I_{\text{im}(A)}) + \text{im}\big(\textstyle\sum_i B_i\big) \subseteq \text{im}(A^{\dagger/2}). \end{aligned}$$

So the claim follows by Fact 13 (using case (2) of the converse). □

3 A Useful Reduction

In the next few lectures, we will study spectral approximations of graphs. Roughly speaking, given a graph G, we would like to find a subgraph H such that $l \cdot L_G \preceq L_H \preceq u \cdot L_G$. That is, the Laplacian of H approximates the Laplacian of G to within a factor u/l. It will be very convenient to apply a reduction to arrive at a simpler problem: instead of trying to approximate L_G, we will try to approximate the identity matrix.

Lemma 6. *Let $G = (V, E)$ be a connected, weighted graph with $n = |V|$ and non-negative edge-weights $x : E \to \mathbb{R}_{\geq 0}$. Then there exist vectors $\{\, w_e \;:\; e \in E \,\} \subset \mathbb{R}^{n-1}$ with $\sum_{e \in E} w_e w_e^T = I$ such that for all non-negative weight vectors $s : E \to \mathbb{R}_{\geq 0}$,*

$$l \cdot I \;\preceq\; \textstyle\sum_{e\in E} s_e w_e w_e^T \;\preceq\; u \cdot I \qquad \Longleftrightarrow \qquad l \cdot L_G \;\preceq\; \underbrace{\textstyle\sum_{e\in E} s_e y_e y_e^T}_{L_H} \;\preceq\; u \cdot L_G,$$

where L_H is the Laplacian of the graph H with weights s. Furthermore, we have $w_e^T w_e = x_e \cdot y_e L_G^{\dagger} y_e$.

Often when we use this lemma, s will be a $\{0, 1\}$-vector, so H will be an unweighted subgraph of G.

Proof. We only prove equivalence of the inequalities involving u. The inequalities involving l are analogous.

Note that $\mathrm{im}(L_H) \subseteq \mathrm{im}(L_G^{\dagger/2})$ since G is connected. So, applying Fact 5,

$$\begin{aligned} L_H \preceq u \cdot L_G &\iff L_G^{\dagger/2} L_H L_G^{\dagger/2} \preceq u \cdot L_G^{\dagger/2} L_G L_G^{\dagger/2} \\ &\iff L_G^{\dagger/2} \big(\textstyle\sum_{e\in E} s_e y_e y_e^T\big) L_G^{\dagger/2} \preceq u \cdot I_{\mathrm{im} L_G} \\ &\iff \textstyle\sum_{e\in E} s_e (L_G^{\dagger/2} y_e)(L_G^{\dagger/2} y_e)^T \preceq u \cdot I_{\mathrm{im} L_G} \end{aligned} \tag{1}$$

Now we use the vectors $\{\, L_G^{\dagger/2} y_e \;:\; e \in E \}$ to derive the desired vectors $\{\, w_e \;:\; e \in E \,\}$.

Let C be a $n \times (n-1)$ matrix whose columns form an orthonormal basis for $\mathrm{im}(L_G) = \mathrm{span}\{\vec{1}\}^{\perp}$. Define

$$w_e = \sqrt{x_e} \cdot C^T L_G^{\dagger/2} y_e \tag{2}$$

Therefore

$$\begin{aligned} \textstyle\sum_{e\in E} x_e y_e y_e^T = L_G \quad &\Rightarrow \quad \textstyle\sum_{e\in E} (\sqrt{x_e} L_G^{\dagger/2} y_e)(\sqrt{x_e} L_G^{\dagger/2} y_e)^T = I_{\mathrm{im} L_G} \\ &\Rightarrow \quad \textstyle\sum_{e\in E} (\sqrt{x_e} C^T L_G^{\dagger/2} y_e)(\sqrt{x_e} C^T L_G^{\dagger/2} y_e)^T = C^T I_{\mathrm{im} L_G} C \\ &\Rightarrow \quad \textstyle\sum_{e\in E} w_e w_e^T = I \end{aligned}$$

Similarly, using Fact 13 part (3),

$$\begin{aligned} \textstyle\sum_{e\in F} s_e (L_G^{\dagger/2} y_e)(L_G^{\dagger/2} y_e)^T \preceq u \cdot I_{\mathrm{im} L_G} \quad &\iff \quad \textstyle\sum_{e\in F} s_e (C^T L_G^{\dagger/2} y_e)(C^T L_G^{\dagger/2} y_e)^T \preceq u \cdot C^T I_{\mathrm{im} L_G} C \\ &\iff \quad \textstyle\sum_{e\in E} s_e w_e w_e^T \preceq u \cdot I. \end{aligned}$$

Combining these equivalences with (1) proves the lemma. □

4 Theorem

Recall the "thin tree" problem. We are given a graph $G = (V, E)$ and $x \in P$, where P is the spanning tree polytope. We would like to find a spanning subtree T of G such that $|\delta_T(U)| \leq O(1) \cdot x(\delta(U))$ for every $U \subseteq V$.

Instead, we will find a "thin forest": a forest satisfying the same inequalities and with at least $n/2$ edges, where $n = |V|$. Similar results were first announced by Goemans (unpublished, 2012). Formally, we prove following theorem:

Theorem 7. *Let $G = (V, E)$ be a connected graph, let $n = |V|$ and assume $|E| \geq 3$. Let $P \subseteq \mathbb{R}^{|E|}$ be its spanning tree polytope. For all $x \in P$, there exists a forest $F \subseteq E$ with $|F| \geq n/2$ such that*

$$L_F \preceq 35 \cdot L_x,$$

where L_x denotes the Laplacian of G with weights x.

By the reduction of the previous section, it suffices to prove the following theorem.

Theorem 8. *Let $G = (V, E)$ be a connected graph, let $n = |V|$ and assume $|E| \geq 3$. Let $P \subseteq \mathbb{R}^{|E|}$ be its spanning tree polytope. Fix any $x \in P$ and suppose that $\{ w_e : e \in E\} \subset \mathbb{R}^{n-1}$ satisfy $\sum_e x_e w_e w_e^T = I$. Then there exists a forest $F \subseteq E$ with $|F| \geq n/2$ such that*

$$\lambda_{\max}\Big(\sum_{e \in F} w_e w_e^T \Big) \preceq 35.$$

This theorem is proven by analyzing the following algorithm.

1. Initialize $A \leftarrow 0$, $F \leftarrow \emptyset$, $u \leftarrow 20$, $\delta = \frac{20}{n-1}$.
2. For $j = 1, \ldots, n/2$:

   INVARIANTS:

   (**a**) *F is acyclic.* (**b**) $\lambda_{\max}(A) < u$. (**c**) $\Phi^u(A) = \mathrm{tr}(uI - A)^{-1} \leq 1/\delta$.

   1. Looping through all edges, find an edge e for which
      i. $F \cup \{e\}$ is acyclic
      ii. $\lambda_{\max}(A + w_e w_e^T) < u + \delta$
      iii. $\Phi^{u+\delta}(A + w_e w_e^T) \leq \Phi^u(A)$
   2. $F \leftarrow F \cup \{e\}$
   3. $A \leftarrow A + w_e w_e^T$
   4. $u \leftarrow u + \delta$

Observe that at the beginning of the algorithm, $\Phi^u(A) = \Phi^{20}(0) = (n-1)/20 = 1/\delta$. At the end of the algorithm, we have a forest F and $A = \sum_{e \in F} w_e w_e^T$ such that

$$\lambda_{\max}(A) < u = 20 + \frac{n\delta}{2} = 20 + \frac{20n}{2(n-1)} = 20 + \frac{10n}{n-1} \leq 35,$$

assuming $n \geq 3$. Thus F satisfies the conditions of the theorem.

It remains to show that the invariants hold and that an edge e will always be found in the algorithm's inner loop. To do so we will require a few preliminary results.

Fact 9. *For $\delta > 0$, $\Phi^u(A) > \Phi^{u+\delta}(A)$.*

Proof. Noting that $\Phi^u(A) := \mathrm{tr}(uI - A)^{-1} = \sum_i (u - \lambda_i(A))^{-1}$, we have

$$\begin{aligned} u - \lambda_i(A) \ < \ u + \delta - \lambda_i(A) \quad &\Rightarrow \quad (u - \lambda_i(A))^{-1} \ > \ (u + \delta - \lambda_i(A))^{-1} \\ &\Rightarrow \quad \textstyle\sum_i (u - \lambda_i(A))^{-1} \ > \ \sum_i (u + \delta - \lambda_i(A))^{-1}. \end{aligned}$$

Therefore Φ is strictly decreasing in δ. □

Fact 10. *Let*

$$M \ := \ ((u+\delta)I - A)^{-1} \qquad \textit{and} \qquad N \ := \ \frac{M^2}{\Phi^u(A) - \Phi^{u+\delta}(A)} + M.$$

Then $M^2/(\Phi^u(A) - \Phi^{u+\delta}(A))$ is positive definite and $N \succ M$.

Proof. By Fact 9, $\Phi^u(A) - \Phi^{u+\delta}(A) > 0$, therefore $M^2/(\Phi^u(A) - \Phi^{u+\delta}(A))$ is positive definite. $N \succ M$ follows since M is positive semidefinite. □

Lemma 11. *Suppose $\lambda_{\max}(A) < u$. For any vector v and positive scalar t, if $v^T N v \le 1/t$ then*

$$\Phi^{u+\delta}(A + tvv^T) \ \le \ \Phi^u(A) \qquad \textit{and} \qquad \lambda_{\max}(A + tvv^T) \ < \ u + \delta$$

This will be proven next time.

A Facts from Linear Algebra

Fact 12. *Let A and B be $n \times n$ symmetric matrices. Then $A \succeq B$ if and only if $x^T Ax \geq x^T Bx$ for all $x \in im(A) + im(B)$*

Proof. Observe that $x^T Ax = (x + x')^T A(x + x')$ for $x' \in \text{null}(A)$. So

$$\begin{aligned} & x^T Ax \geq x^T Bx && \forall x \in \text{im}(A) + \text{im}(B) \\ \iff & (x + x')^T A(x + x') \geq (x + x')^T B(x + x') && \forall x \in \text{im}(A) + \text{im}(B), \forall x' \in \text{null}(A) \cap \text{null}(B) \\ \iff & y^T Ay \geq y^T By && y = x + x',\ \forall x \in \text{im}(A) + \text{im}(B),\ \forall x' \in \text{null}(A) \cap \text{null}(B) \\ \iff & y^T Ay \geq y^T By && y \in \mathbb{R}^n \end{aligned}$$

because $(\text{null}(A) \cap \text{null}(B))^\perp = \text{null}(A)^\perp + \text{null}(B)^\perp = \text{im}(A) + \text{im}(B)$, therefore $[\text{im}(A) + \text{im}(B)] + [\text{null}(A) \cap \text{null}(B)] = \mathbb{R}^n$. □

Fact 13. *Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices and $C \in \mathbb{R}^{n\times m}$, and $m \leq n$. If A, B are symmetric then*

$$B \succeq A \quad \Longrightarrow \quad C^T BC \succeq C^T AC.$$

The converse holds if

1. *C is square and nonsingular*
2. *C is symmetric and $im(A) + im(B) \subseteq im(C)$ or equivalently $null(A) \cap null(B) \supseteq null(C)$*
3. *C has orthonormal columns and $im(A) + im(B) \subseteq im(C)$ or equivalently $null(A) \cap null(B) \supseteq null(C^T)$*

Proof. $B \succeq A \iff A - B \succeq 0$. Therefore there exists a matrix V such that $A - B = VV^T$. Then

$$C(A - B)C^T = CVV^T C^T = (CV)(CV)^T$$

which shows $C(A - B)C^T$ is positive semi definite. We now consider the three cases in the converse.

(1) Since C is non-singular, C^{-1} exists and

$$C^T BC \succeq C^T AC \quad \Longrightarrow \quad (C^{-1}C)^T B(CC^{-1}) \succeq (C^{-1}C)^T A(CC^{-1}) \quad \Longrightarrow \quad B \succeq A.$$

(2) Since C is symmetric, $C^\dagger$ is also symmetric and hence $C^\dagger C = CC^\dagger = I_{\text{im}(C)}$.

$$\begin{aligned} C^T BC \succeq C^T AC & \Longrightarrow (C^\dagger C)^T B(CC^\dagger) \succeq (C^\dagger C)^T A(CC^\dagger) \\ & \Longrightarrow I_{\text{im}(C)} BI_{\text{im}(C)} \succeq I_{\text{im}(C)} AI_{\text{im}(C)}. \\ & \Longrightarrow x^T I_{\text{im}(C)} BI_{\text{im}(C)} x \geq x^T I_{\text{im}(C)} AI_{\text{im}(C)} x && \forall x \in \mathbb{R}^n \\ & \Longrightarrow x^T Bx \geq x^T Ax && \forall x \in \text{im}(C) \\ & \Longrightarrow x^T Bx \geq x^T Ax && \forall x \in \text{im}(A) + \text{im}(B) \\ & \Longrightarrow B \succeq A \end{aligned}$$

The last implication comes from Fact 12.

(3) Complete C to a square, orthogonal matrix $\bar{C}$ by adding $n - m$ new columns. That is, $\bar{C} = [C, D]$ and $\bar{C}^T \bar{C} = \bar{C}\bar{C}^T = I$. (This can be done, for example, by the Gram-Schmidt orthogonalization process.)

Then

$$
\begin{aligned}
B \succeq A &\overset{\text{by (1)}}{\iff} \bar{C}^T B \bar{C} \succeq \bar{C} A \bar{C} \\
&\iff x^T \bar{C}^T B \, \bar{C} x \geq x^T \bar{C}^T A \, \bar{C} x && \forall x \in \mathbb{R}^n \\
&\iff x^T \begin{pmatrix} C^T BC & C^T BD \\ D^T BC & D^T BD \end{pmatrix} x \geq x^T \begin{pmatrix} C^T AC & C^T AD \\ D^T AC & D^T AD \end{pmatrix} x && \forall x \in \mathbb{R}^n \\
&\overset{\text{(a)}}{\iff} x^T \begin{pmatrix} C^T BC & 0 \\ 0 & 0 \end{pmatrix} x \geq x^T \begin{pmatrix} C^T AC & 0 \\ 0 & 0 \end{pmatrix} x && \forall x \in \mathbb{R}^n \\
&\iff y^T C^T BC y \geq y^T C^T AC y && \forall y \in \mathbb{R}^m \\
&\iff C^T BC \succeq C^T AC
\end{aligned}
$$

In (a), we use the fact that $\text{im}(D) = \text{null}(C^T) \subseteq \text{null}(A) \cap \text{null}(B)$, Therefore $AD = BD = 0$. This shows every term in the matrix becomes 0 except the entries in the top left. □