Optimal Auctions

Game Theory Course:
Jackson, Leyton-Brown & Shoham

Optimal Auctions

- So far we have considered efficient auctions.
- What about maximizing the seller's revenue?
  - she may be willing to risk failing to sell the good.
  - she may be willing sometimes to sell to a buyer who didn't make the highest bid

Optimal auctions in an independent private values setting

- private valuations
- risk-neutral bidders
- each bidder i's valuation independently drawn from a strictly increasing cumulative density function $F_i(v)$ with a pdf $f_i(v)$ that is continuous and bounded below
  - Allow $F_i \neq F_j$: asymmetric auctions
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The auction that maximizes the seller's expected revenue subject to (ex post, interim) individual rationality and Bayesian incentive compatibility for the buyers is an optimal auction.

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- Which reserve price R maximizes expected revenue?

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  - sale at price R if one bid above reserve and other below - happens with probability $2(1-R)R$ and revenue $= R$
  - sale at second highest bid if both bids above reserve - happens with probability $(1-R)^2$ and revenue $= E[\min v_i | \min v_i \geq R] = \frac{1+2R}{3}$

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- Maximizing: $0 = 2R - 4R^2$, or $R = \frac{1}{2}$.

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  - increase price when one bidder has low value other high: happens with probability 1/2
- Like adding another bidder: increasing competition in the auction.

Designing optimal auctions

Definition (virtual valuation)

Bidder i's virtual valuation is $\psi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$.

Let us assume this is increasing in v_i (e.g., for a uniform distribution it is $2v_i - 1$).

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Definition (bidder-specific reserve price)

Bidder i's bidder-specific reserve price r_i^* is the value for which $\psi_i(r_i^*) = 0$.

Myerson's Optimal Auctions

Theorem (Myerson (1981))

The optimal (single-good) auction in terms of a direct mechanism: The good is sold to the agent $i = \arg\max_i \psi_i(\hat{v}_i)$*, as long as* $v_i \geq r_i^*$*. If the good is sold, the winning agent* i *is charged the smallest valuation that he could have declared while still remaining the winner:* $\inf\{v_i^* : \psi_i(v_i^*) \geq 0$ *and* $\forall j \neq i,\ \psi_i(v_i^*) \geq \psi_j(\hat{v}_j)\}$.

Myerson's Optimal Auctions

Corollary (Myerson (1981))

In a symmetric setting, the optimal (single-good) auction is a second price auction with a reserve price of r^* *that solves* $r^* - \frac{1-F(r^*)}{f(r^*)} = 0$.

Analyzing optimal auctions

Optimal Auction:

- winning agent: $i = \arg\max_i \psi_i(\hat{v}_i)$, as long as $v_i \geq r_i^*$.
- i is charged the smallest valuation that he could have declared while still remaining the winner, $\inf\{v_i^* : \psi_i(v_i^*) \geq 0$ and $\forall j \neq i,\ \psi_i(v_i^*) \geq \psi_j(\hat{v}_j)\}$.

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- Is this VCG?
  - No, it's not efficient.
- How should bidders bid?
  - it's a second-price auction with a reserve price, held in virtual valuation space.
  - neither the reserve prices nor the virtual valuation transformation depends on the agent's declaration
  - thus the proof that a second-price auction is dominant-strategy truthful applies here as well.

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- winning agent: $i = \arg\max_i \psi_i(\hat{v}_i)$, as long as $v_i > r_i^*$.
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- Why does this work?
  - reserve prices are like competitors: increase the payments of winning bidders
  - the virtual valuations can increase the impact of weak bidders' bids, making them more competitive.
  - bidders with higher expected valuations bid more aggressively