STOCHASTIC LOCAL SEARCH
FOUNDATIONS AND APPLICATIONS

Generalised Local Search Machines

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Outline

The Basic GLSM Model

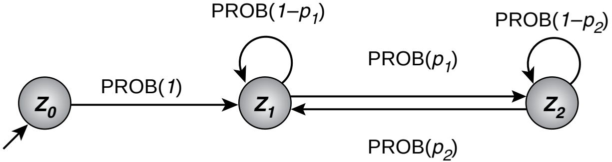
Many high-performance SLS methods are based on combinations of *simple (pure) search strategies* (*e.g.*, ILS, MA).

These hybrid SLS methods operate on two levels:

- **lower level:** execution of underlying simple search strategies
- **higher level:** activation of and transition between lower-level search strategies.

Key idea underlying Generalised Local Search Machines:
Explicitly represent higher-level search control mechanism in the form of a *finite state machine*.

Example: Simple 3-state GLSM



- States z_0, z_1, z_2 represent simple search strategies, such as Random Picking (for initialisation), Iterative Best Improvement and Uninformed Random Walk.
- PROB(p) refers to a probabilistic state transition with probability p after each search step.

Generalised Local Search Machines (GLSMs)

- States $\cong$ simple search strategies.
- State transitions $\cong$ search control.

- GLSM $\mathcal{M}$ starts in initial state.
- In each iteration:
  - $\mathcal{M}$ executes one search step associated with its current state z;
  - $\mathcal{M}$ selects a new state (which may be the same as z) in a nondeterministic manner.

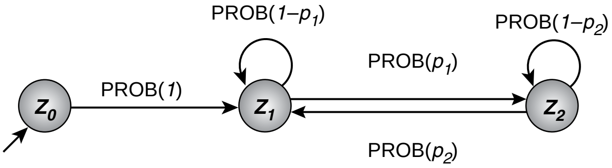
- $\mathcal{M}$ terminates when a given termination criterion is satisfied.

Formal definition of a GLSM

A *Generalised Local Search Machine* is defined as a tuple $\mathcal{M} := (Z, z_0, M, m_0, \Delta, \sigma_Z, \sigma_\Delta, \tau_Z, \tau_\Delta)$ where:

- Z is a set of states;
- $z_0 \in Z$ is the *initial state*;
- M is a set of *memory states* (as in SLS definition);
- m_0 is the *initial memory state* (as in SLS definition);
- $\Delta \subseteq Z \times Z$ is the *transition relation*;
- σ_Z and σ_Δ are sets of *state types* and *transition types*;
- $\tau_Z : Z \mapsto \sigma_Z$ and $\tau_\Delta : \Delta \mapsto \sigma_\Delta$ associate every state z and transition (z, z') with a state type $\sigma_Z(z)$ and transition type $\tau_\Delta((z, z'))$, respectively.

Example: Simple 3-state GLSM (formal definition)



- $Z := \{z_0, z_1, z_2\}$; z_0 = initial machine state
- no memory ($M := \{m_0\}$; m_0 = initial and only memory state)
- $\Delta := \{(z_0, z_1), (z_1, z_2), (z_1, z_1), (z_2, z_1), (z_2, z_2)\}$
- $\sigma_Z := \{z_0, z_1, z_2\}$
- $\sigma_\Delta := \{\text{PROB}(p) \mid p \in \{1, p_1, p_2, 1 - p_1, 1 - p_2\}\}$
- $\tau_Z(z_i) := z_i, \quad i \in \{0, 1, 2\}$
- $\tau_\Delta((z_0, z_1)) := \text{PROB}(1)$, $\tau_\Delta((z_1, z_2)) := \text{PROB}(p_1)$, ...

Example: Simple 3-state GLSM (semantics)

- Start in initial state z_0, memory state m_0 (never changes).
- Perform one search step according to search strategy associated with state type z_0 (*e.g.*, random picking).
- With probability 1, switch to state z_1.
- Perform one search step according to state z_1;
  switch to state z_2 with probability p_1,
  otherwise, remain in state z_1.
- In state z_2, perform one search step according to z_2;
  switch back to state z_1 with probability p_2,
  otherwise, remain in state z_2.

$\rightsquigarrow$ After one z_0 step (initialisation), repeatedly and nondeterministically switch between phases of z_1 and z_2 steps until termination criterion is satisfied.

Note:

- *States types* formally represent (subsidiary) search strategies, whose definition is not part of the GLSM definition.
- *Transition types* formally represent mechanisms used for switching between GLSM states.
- Multiple states / transitions can have the same type.
- σ_Z, σ_Δ should include only state and transition types that are actually used in given GLSM ('no junk').
- Not all states in Z may actually be reachable when running a given GLSM.
- *Termination condition* is not explicitly captured GLSM model, but considered part of the execution environment.

GLSM Semantics

Behaviour of a GLSM is specified by *machine definition* +
run-time environment comprising specifications of

- state types,
- transition types;
- problem instance to be solved,
- search space,
- solution set,
- neighbhourhood relations for subsidiary SLS algorithms;
- termination predicate for overall search process.

Run GLSM $\mathcal{M}$:

set *current machine state* to z_0; set *current memory state* to m_0;

While *termination criterion* is not satisfied:

- perform *search step* according to type of current machine state; this results in a new *search position*
- select *new machine state* according to *types of transitions* from *current machine state*, possibly depending on *search position* and *current memory state*; this may change the *current memory state*

Note:

- The *current search position* is only changed by the subsidiary search strategies associated with states, *not* as side-effect of machine state transitions.
- The *machine state* and *memory state* are only changed by state-transitions, *not* as side-effect of search steps. (Memory state is viewed as part of higher-level search control.)
- The operation of $\mathcal{M}$ is uniquely characterised by the evolution of *machine state*, *memory state* and *search position* over time.

GLSMs are factored representations of SLS strategies:

- Given GLSM represents the way in which *initialisation* and *step function* of a hybrid SLS method are composed from respective functions of *subsidiary component SLS methods*.

- When modelling hybrid SLS methods using GLSMs, *subsidiary SLS methods* should be as simple and pure as possible, leaving *search control* to be represented explicitly at the GLSM level.

- *Initialisation* is modelled using *GLSM states* (advantage: simplicity and uniformity of model).

- *Termination of subsidiary search strategies* are often reflected in *conditional transitions* leaving respective GLSM states.

State, Transition and Machine Types

In order to completely specify the search method represented by a given GLSM, we need to define:

- the GLSM model (states, transitions, ...);
- the search method associated with each *state type*, *i.e.*, step functions for the respective subsidiary SLS methods;
- the semantics of each *transition type*, *i.e.*, under which conditions respective transitions are executed, and how they effect the memory state.

State types

- State type semantics are often most conveniently specified procedurally (see algorithm outlines for 'simple SLS methods' from Chaper 2).

- *initialising state type* = state type τ for which search position after one τ step is independ of search position before step.

  initialising state = state of initialising type.

- *parametric state type* = state type τ whose semantics depends on memory state.

  parametric state = state of parametric type.

Transitions types (1)

- *Unconditional deterministic transitions* – type *DET*:
  - executed always and independently of memory state or search position;
  - every GLSM state can have at most one outgoing DET transition;
  - frequently used for leaving initialising states.
- *Conditional probabilistic transitions* – type *PROB(p)*:
  - executed with probability p, independently of memory state or search position;
  - probabilities of PROB transitions leaving any given state must sum to one.

Note:

- DET transitions are a special case of PROB transitions.
- Given GLSM $\mathcal{M}$ any state that can be reached from initial state z_0 by following a chain of PROB(p) transitions with $p > 0$ with eventually be reached with arbitrarily high probability in any sufficiently long run of $\mathcal{M}$.
- In any state z with a PROB(p) self-transition (z, z) with $p > 0$, the number of GLSM steps before leaving z is distributed geometrically with mean and variance $1/p$.

Transitions types (2)

- *Conditional probabilistic transitions* – type *CPROB(C, p)*:
  - executed with probability proportional to p iff *condition predicate C* is satisfied;
  - all CPROB transitions from the current GLSM state whose condition predicates are not satisfied are *blocked*, *i.e.*, cannot be executed.

Note:

- Special cases of CPROB(C, p) transitions:
  - PROB(p) transitions;
  - *conditional deterministic transitions*, type *CDET(C)*.
- Condition predicates should be efficiently computable (ideally: $\leq$ linear time w.r.t. size of given problem instance).

Commonly used simple condition predicates:

$\top$	always true
count(k)	total number of GLSM steps $\geq k$
countm(k)	total number of GLSM steps modulo $k = 0$
scount(k)	number of GLSM steps in current state $\geq k$
scountm(k)	number of GLSM steps in current state modulo $k = 0$
lmin	current candidate solution is a local minimum w.r.t. the given neighbourhood relation
evalf(y)	current evaluation function value $\leq y$
noimpr(k)	incumbent candidate solution has not been improved within the last k steps

All based on local information; can also be used in negated form.

Transition actions:

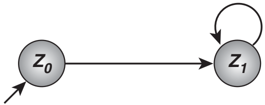
- Associated with individual transitions; provide mechanism for modifying current memory states.
- Performed whenever GLSM executes respective transition.
- Modify memory state only, *cannot* modify GLSM state or search position.
- Have read-only access to search position and can hence be used, *e.g.*, to memorise current candidate solution.
- Can be added to any of the previously defined transition types.

Machine types:

Capture *structure of search control mechanism*, obtained from abstracting from state and transition types of GLSMs.

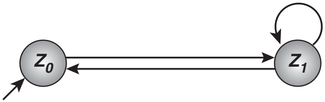
- *1-state machines*:
  - simplest machine type, single initialising state only;
  - realises iterated sampling processes, such as Uninformed Random Picking.
- *1-state+init machines*:
  - one initialising + one working state;
  - good model for many simple SLS methods.

- *sequential 1-state machines*:



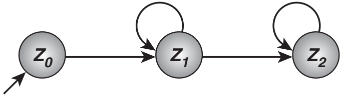
  - visit initialising state z_0 only on once.

- *alternating 1-state+init machines*:



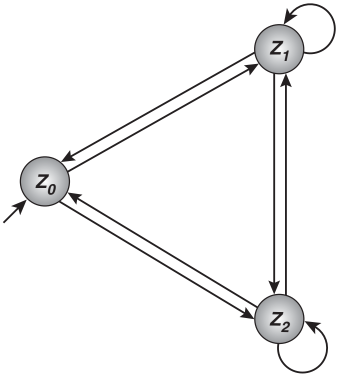
  - may visit initialising state z_0 multiple times;
  - good model for simple SLS methods with restart mechanism.

- *2-state+init sequential machines*:
  - one initialising state (visited only once), two working states;



  - any search trajectory can be partitioned into three phases: one initialisation step, a sequence of z_1 steps and a sequence of z_2 steps.

- *2-state+init alternating machines*:
  - one initialising state, two working states;
  - arbitrary transitions between any states are possible.



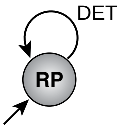
Generalisations:

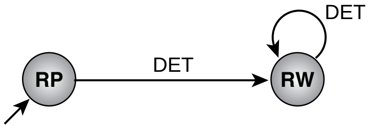
- *k-state+init sequential machines*:
  - one initialising state (visited only once), k working states;
  - every search trajectory consists of $1+k$ phases.

- *k-state+init alternating machines*:
  - one initialising state, k working states;
  - arbitrary transitions between states;
  - may have multiple initialising states (*e.g.*, to realise alternative restart mechanisms).

Modellig SLS Methods Using GLSMs

Uninformed Picking and Uninformed Random Walk





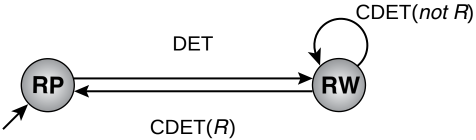
procedure *step-RP*(π, s)
  input: *problem instance* $\pi \in \Pi$,
    candidate solution $s \in S(\pi)$
  output: *candidate solution* $s \in S(\pi)$

  $s' := selectRandom(S)$;
  return s'
end *step-RP*

procedure *step-RW*(π, s)
  input: *problem instance* $\pi \in \Pi$,
    candidate solution $s \in S(\pi)$
  output: *candidate solution* $s \in S(\pi)$

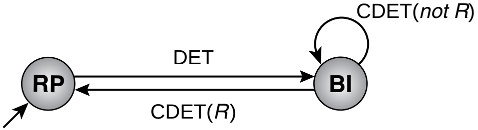
  $s' := selectRandom(N(s))$;
  return s'
end *step-RW*

Uninformed Random Walk with Random Restart



R = restart predicate, *e.g.*, countm(k)

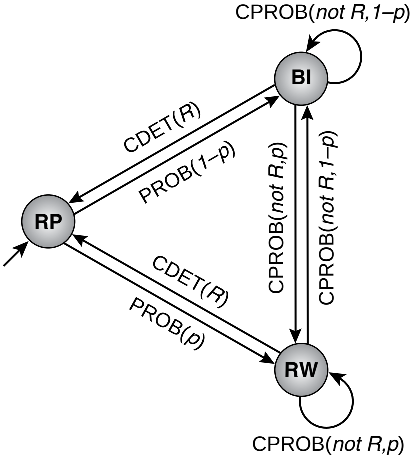
Iterative Best Improvement with Random Restart



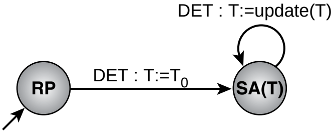
procedure *step-BI*(π, s)
  input: *problem instance* $\pi \in \Pi$, *candidate solution* $s \in S(\pi)$
  output: *candidate solution* $s \in S(\pi)$

  $g^* := \min\{g(s') \mid s' \in N(s)\};$
  $s' := selectRandom(\{s' \in N(s) \mid g(s') = g^*\});$
  return s'
end *step-BI*

Randomised Iterative Best Improvement with Random Restart

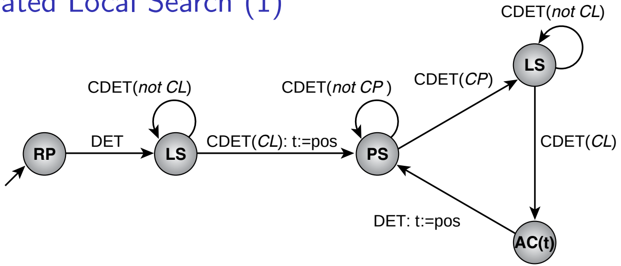


Simulated Annealing



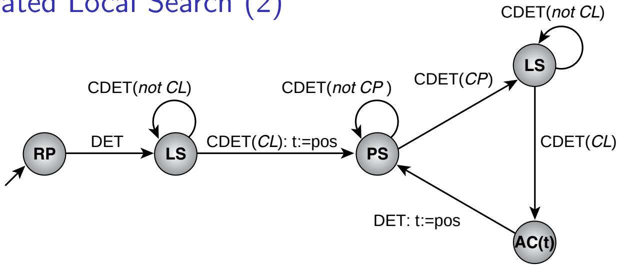
- Note the use of transition actions and memory for temperature T.
- The parametric state SA(T) implements probabilistic improvement steps for given temperature T.
- The initial temperature T_0 and function *update* implement the annealing schedule.

Iterated Local Search (1)



- The acceptance criterion is modelled as a state type, since it affects the search position.
- Note the use of transition actions for memorising the current candidate solution (pos) at the end of each local search phase.
- Condition predicates *CP* and *CL* determine the end of perturbation and local search phases, respectively; in many ILS algorithms, $CL :=$ lmin.

Iterated Local Search (2)



procedure *step-AC*(π, s, t)
  input: *problem instance* $\pi \in \Pi$,
    candidate solution $s \in S(\pi)$
  output: *candidate solution* $s \in S(\pi)$

  if $C(\pi, s, t)$ **then**
    return s
  else
    return t
  end
end *step-AC*

Ant Colony Optimisation (1)

- General approach for modelling population-based SLS methods, such as ACO, as GLSMs:

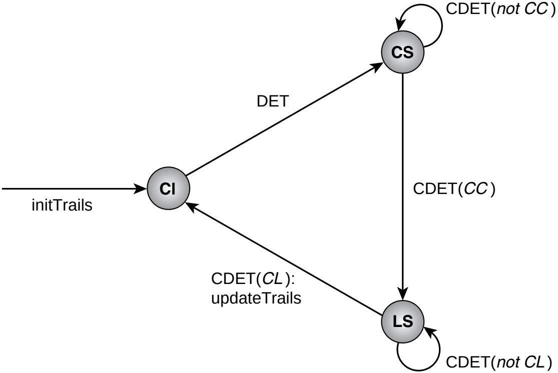
  Define search positions as *sets of candidate solutions*; search steps manipulate some or all elements of these sets.

  Example: In this view, Iterative Improvement (II) applied to a population *sp* in each step performs one II step on each candidate solution from *sp* that is not already a local minimum.

  (Alternative approaches exist.)

- Pheromone levels are represented by memory states and are initialised and updated by means of transition actions.

Ant Colony Optimisation (2)



- The condition predicate *CC* determines the end of the construction phase.
- The condition predicate *CL* determines the end of the local search phase; in many ACO algorithms, $CL :=$ lmin.

Extensions of the Basic GLSM Model

The basic GLSM model can be generalised and extended in various rather straightforward ways, such as:

- Co-operative GLSM models
- Learning GLSM models
- Evolutionary GLSM models
- Continuous GLSM models

Note: So far, these extensions remain mostly unexplored — lots of opportunities for interesting research!

Co-operative GLSM models

- **Key idea:** Apply multiple GLSMs simultanously to the same problem instance
- Naturally captures population-based SLS approaches.
- *Homogeneous co-operative GLSM models:*
  Population of identical GLSMs; equivalent to performing multiple independent runs of the respective SLS method.
- *Heterogenous co-operative GLSM models:*
  Population of different GLSMs; model *algorithm portfolios*.

Co-operative GLSM models with communication

- GLSMs in population exchange information about their search trajectories, *e.g.*, via message passing or blackboard mechanism.
- Communication can be modelled via shared memory state or special transition actions (*e.g.*, *send*, *receive*).
- These models are naturally suited for representing population-based algorithms that use communication between individual search agents, such as ACO.

Learning via dynamic transition probabilities

- **Key idea:** In a GLSM with probabilistic transitions, let transition probabilities evolve over time to adaptively optimise search control strategy.
- Can build on concepts from learning automata theory.

- *Single-instance learning:*
  Optimise control strategy on one problem instance during search process.
- *Multi-instance learning:*
  Adapt control strategies to features common to a class of problem instances.

- Transition probabilities can be adapted via external mechanism or via specialised transition actions.

Evolutionary GLSM models

- **Key idea:** Achieve learning/adaptation in co-operative GLSM models by varying number or type of individual GLSMs over time.
- Distinction between *single-* and *multi-instance learning* as before; similar mechanisms for controlling adaptation process.
- Can easily model, for example, self-optimising portfolios of SLS algorithms.
- Further extensions:
  - support mutation / recombination operations on GLSMs;
  - additionally support learning in individual GLSMs $\rightsquigarrow$ evolving ensembles of dynamic GLSMs;
  - include communication between GLSMs in population.

Continuous GLSM models

- *Note:* Many previously discussed hybrid SLS methods can be extended to continuous optimisation problems and give rise to high-performance algorithms for solving these.
- The main feature of the GLSM model, namely its clear distinction between *lower-level, simple search strategies* and *higher-level search control*, equally applies to continuous SLS algorithms.
- **Key idea:** Model complex continuous SLS methods by using continuous optimisation procedures as subsidiary local search strategies.

Note: The GLSM model is well-suited for modelling algorithms for *hybrid combinatorial problems* that involve discrete as well as continuous solution components.