Lecture Stat 302
Introduction to Probability - Slides 8

AD

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Expected Value

- The expectation of X, denoted $E(X)$, is defined as

$$E(X) = \sum_{i=1}^{\infty} x_i p(x_i) = \sum_{x:p(x)>0} x \ p(x)$$

- It corresponds to a weighted average of the possible values that X can take on. Intuitively if you consider an infinite number of experiments then the average will be $E(X)$.
- **Example**: Assume you have a fair die and X is the number rolled then $p(i) = \frac{1}{6}$ for $i = 1, 2, ..., 6$ and

$$\begin{aligned} E(X) &= 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} \\ &= \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = \frac{7}{2} \end{aligned}$$

Example: Expected Payout of Roulette Wheel

- *Red/Black*: We have $P(X_1 = 1) = \frac{18}{38}$ and $P(X_1 = -1) = \frac{20}{38}$ so

$$E(X_1) = 1 \times \frac{18}{38} + (-1) \times \frac{20}{38} = -\frac{1}{19} \approx -0.053$$

- *Straight up*: We have $P(X_2 = 35) = \frac{1}{38}$ and $P(X_2 = -1) = \frac{37}{38}$ so

$$E(X_2) = 35 \times \frac{1}{38} + (-1) \times \frac{37}{38} = -\frac{1}{19}.$$

- *Split bet*: We have $P(X_3 = 17) = \frac{2}{38}$ and $P(X_3 = -1) = \frac{36}{38}$ so

$$E(X_3) = 17 \times \frac{2}{38} + (-1) \times \frac{36}{38} = -\frac{1}{19}.$$

- *Street bet*: We have $P(X_4 = 11) = \frac{3}{38}$ and $P(X_4 = -1) = \frac{35}{38}$ so

$$E(X_4) = 11 \times \frac{3}{38} + (-1) \times \frac{35}{38} = -\frac{1}{19}.$$

Example: Tramways in Melbourne

- In Melbourne (Australia), no one collects fare when you board a tram and no one checks systematically whether you have a ticket. Only occasionally, inspectors arrive to check tickets and give a fine to any ticketless riders. A ticket costs 2$, a fine is 30$ and the probability of meeting inspectors 5%. Assume you are interested in minimizing the average cost, should you buy tram tickets or not?
- Let C being the cost. If you buy a ticket each time you board a tram then the cost is $C = 2\$$ with proba. 1 (i.e. there is no uncertainty, i.e. $P(C = 2) = 1$) so the average cost is

$$E(C) = 2 \times 1 = 2\$.$$

If you never buy tram tickets then the cost is $C = 0\$$ if you don't get caught and $C = 30\$$ if you get caught so $P(C = 0) = 0.95$ and $P(C = 30) = 0.05$ so

$$E(C) = 0 \times 0.95 + 30 \times 0.05 = 1.50\$$$

So as to minimize the average cost, do not buy tram tickets.

Example: Lottery

- You are playing the lottery 6/49. A ticket costs 2$ and assume you win 10$ for 3 correct numbers, 250$ for 4 correct numbers, 10,000$ for 5 correct numbers and 2,000,000$ for 6 correct numbers. What is the average payout X of lottery?
- So we have

$$X = \begin{cases} \text{-2} & \text{if not even 3 numbers} \\ 10 & \text{if 3 correct numbers} \\ 250 & \text{if 4 correct numbers} \\ 10{,}000 & \text{if 5 correct numbers} \\ 2{,}000{,}000 & \text{if 6 correct numbers} \end{cases}$$

Example: Lottery

- We have computed all these probability before.... see Slides 4!
- We have

$$
\begin{aligned}
E(X) &= -2\times 0.9814+10\times 0.0176+250\times 9.686.\ 10^{-4} \\
&\quad +10000\times 1.844.\ 10^{-5}+2000000\times 7.151\times 10^{-8} \\
&= -2+0.176+0.242+0.184+0.143 \\
&= -1.255
\end{aligned}
$$

- This is much worse than playing roulette.

Example: Loan Workout Project

- You are loan officers of CIBC.
- CIBC has a commercial loan with a full value of $4,000,000 outstanding to John Smith's entrepreneurial venture.
- His business is not going well and he is not able to pay interest on his loan.
- What are you going to do as loan officers?

Example: Loan Workout Project

- CIBC has to decide whether to foreclose on the loan or to work out a new schedule of payments.
- If CIBC forecloses on John's loan, it will only recover a foreclosure value of $2,100,000.
- If CIBC attempts a workout, and John's venture succeeds, then the bank will be repaid the full value of $4,000,000.
- If CIBC attempts a workout, and John's venture fails, then the bank will only recover a default value of $250,000.
- How do you decide which one to choose? Foreclosure or workout?

Example: Loan Workout Project

- One way to decide consists of comparing the foreclosure value to the expected value of workout.
- If foreclosure value $>$ expected value of workout then choose foreclosure, otherwise choose workout.
- The expected value of workout is

$$E\left(Workout\right) = 4,000,000 \times P\left(Success\right) + 250,000 \times P\left(Fail\right).$$

- Typically $P\left(Success\right)$ and $P\left(Fail\right)$ are obtained using historical data on past borrowers, certain information about John Smith, current economic conditions and conditional Probability and Bayes' Theorem.
- Assuming $P\left(Success\right) = 0.4 = 1 - P\left(Fail\right)$, then $E\left(Workout\right) = 1,750,000 < 2,100,000$ so the loan officers go for foreclosure.

Example: Who wants to be a millionnaire?

- You have won \$64,000 and are facing a tough \$125,000 question. This means that you will receive \$125,000 if you answer the question correctly. But you will receive only \$32,000 if your answer is wrong. Also, you can walk out with \$64,000 if you decide not to answer the question. What would be the best strategy to take?
- The \$125,000 question is the following. Which city is the birthplace of your 302 professor?

A- Bordeaux	B- Cognac
C- Nice	D- Paris

- Suppose that you decide to answer the question. What is the expected value of the outcome? Would you answer the question or do you walk out with \$64,000?

Example: Who wants to be a millionnaire?

- You have expected value of the outcome

$$\underbrace{\frac{1}{4}}_{\text{Proba correct}} \times 125,000 + \underbrace{\frac{3}{4}}_{\text{Proba incorrect}} \times 32,000 = 55,250 < 64,000$$

  whereas if you walk out now you have 64,000$.
- It seems perhaps more sensible to walkout although you might like taking risks and stay in.

Example: Who wants to be a millionnaire?

- Now assume you can use the 50:50 option to get rid of two of the potential answers, so that

  A- Bordeaux   B- Cognac

  Suppose that you decide to answer the question. What is the expected value of the outcome? Would you answer the question or would you walk out with \$64,000?

- We have expected value

$$\underbrace{\frac{1}{2}}_{\text{Proba correct}} \times 125,000 + \underbrace{\frac{1}{2}}_{\text{Proba incorrect}} \times 32,000 = 78,500 > 64,000.$$

- It might be reasonable to answer although a risk averse person will definitely not answer!

Expected Utility

- In the previous example, we have discussed cases where we compute the expected cost, payout or price under different situations so as to make a decision: buy/don't buy a ticket, answer/don't answer question etc.
- In these cases, the cost, payout, reward is what we call a utility.
- Different people have typically different utility functions; e.g. some people will always buy a tram ticket and their utility is not the price.
- This also explains why many people play lottery even if the expected payout is terrible. They value winning a large amount of money much more than spending 2$ a week.

Expected Utility: Dating

- You fancy a person but you feel that he/she might be out of your league. Your utility is your happiness whose reference level is zero. You'd like to maximize your average happiness. If you could date this person, then your happiness would increase by say $1,000$. If you ask her/him out and you got rejected, then your happiness will plunge to -100. Given the probability that the person accepts going out with you is 0.1, should you try to ask her/him out?
- If you don't ask her/him out, then deterministically $P(H = 0) = 1$ so $E(H) = 0$.
- If you ask her/him out then $P(H = 1000) = 0.1$ and $P(H = -100)$ so

$$E(H) = 1000 \times 0.1 + (-100) \times 0.9 = 10 > 0$$

so you should try.

Expectation of a Function of a Random Variable

- Given a random variable X, we can define a new random variable $Y = g(X)$.
- From the probability mass function (pmf) of X, we can obtain the pmf of Y.
- **Example:** Assume you have X taking values in $\{-2, -1, 0, 1, 2\}$ and consider $Y = X^2$. Clearly Y takes values in $\{0, 1, 4\}$ and we have

$$\begin{aligned} P(Y=4) &= P(X=2) + P(X=-2), \\ P(Y=1) &= P(X=1) + P(X=-1), \\ P(Y=0) &= P(X=0). \end{aligned}$$

The expected value of Y is

$$E[Y] = 0 \times P(Y=0) + 1 \times P(Y=1) + 4 \times P(Y=4).$$

Expectation of a Function of a Random Variable

- Let X be a discrete r.v. taking values $\{x_i\}$ with pmf $\{p_X(x_i)\}$ and define $Y = g(X)$.
- **Proposition**. We have

$$E(Y) = E(g(X)) = \sum_i g(x_i)\, p_X(x_i).$$

- The proof follows from

$$\begin{aligned} E(Y) &= \sum_j y_j P(Y = g(X) = y_j) = \sum_j y_j \sum_{i: g(x_i) = y_j} p_X(x_i) \\ &= \sum_j \sum_{i: g(x_i) = y_j} y_j p_X(x_i) \\ &= \sum_i g(x_i)\, p_X(x_i) \end{aligned}$$

Consequences

- Consider $Y = aX + b$ then

$$\begin{aligned} E(Y) &= E(aX+b) = \sum_i (ax_i + b)\, p_X(x_i) \\ &= a\sum_i x_i p_X(x_i) + b\sum_i p_X(x_i) \\ &= aE(X) + b \end{aligned}$$

- Consider $Y = g(X)$ and $Z = f(X)$ then

$$\begin{aligned} E(Y+Z) &= \sum_i (g(x_i) + f(x_i))\, p_X(x_i) \\ &= \sum_i g(x_i)\, p_X(x_i) + \sum_i f(x_i)\, p_X(x_i) \\ &= E(Y) + E(Z). \end{aligned}$$