HW 1

Due 23:59 May 13, 2019

CS ID 1: ____________

CS ID 2: ____________

Instructions:

1. Do not change the problem statements we are giving you. Simply add your solutions by editing this latex document.
2. Take as much space as you need for each problem. You'll tell us where your solutions are when you submit your paper to gradescope.
3. Export the completed assignment as a PDF file for upload to gradescope.
4. On gradescope, upload only **one** copy per partnership. (Instructions for uploading to gradescope will be posted on the HW1 page of the course website.)

Academic Conduct: I certify that my assignment follows the academic conduct rules for of CPSC 221 as outlined on the course website. As part of those rules, when collaborating with anyone outside my group, (1) I and my collaborators took no record but names away, and (2) after a suitable break, my group created the assignment I am submitting without help from anyone other than the course staff.

1. (2 points) Using 140 characters or less, post a synopsis of your favorite movie to the course piazza space under the "HW1 tell me something!" notice, so that your post is visible to everyone in the class, and tagged by #HW1num1. Also, use Piazza's code-formatting tools to write a *private* post to course staff that includes at least 5 lines of code. It can be code of your own or from a favorite project—it doesn't even have to be syntactically correct—but it must be formatted as a code block in your post, and also include the tag #HW1num1. (Hint: Check http://support.piazza.com/customer/portal/articles/1774756-code-blocking). Finally, please write the 2 post numbers corresponding to your posts here:

Favorite Movie Post (Public) number:	
Formatted Code Post (Private) number:	

2. (16 points) In this problem, you will be a math detective! Some of the symbols and functions may not be familiar to you, but with a little digging, reading, and observing, you'll be able to figure them out. Your task is to simplify each of the following expressions as much as possible, **without using a calculator (either hardware or software)**. Do not approximate. Express all rational numbers as improper fractions. You may assume that n is an integer greater than 1. Show your work in the space provided, and write your final answer in the box.

   (a) i. $\displaystyle\sum_{k=6}^{n}(5+\sqrt{4^k})$

Answer for (a.i):	$5n + 2^{n+1} - 89$

$$
\begin{aligned}
&= \sum_{k=6}^{n} 5 + \sum_{k=6}^{n} 2^k \\
&= (n-6+1)\times 5 + \sum_{k=0}^{n} 2^k - \sum_{k=0}^{5} 2^k \\
&= 5(n-5) + (2^{n+1}-1) - 63 \\
&= 5n + 2^{n+1} - 89
\end{aligned}
$$

ii. $\displaystyle\prod_{i=8}^{n} \frac{2i^2}{(i^2-1)}$

Answer for (a.ii):	$2^{n-7} \times \dfrac{8 \cdot n}{7 \cdot (n+1)}, n \geq 8$

$$
\begin{aligned}
&= 2^{n-8+1} \times \prod_{i=8}^{n} \frac{i^2}{(i+1)(i-1)} \\
&= 2^{n-7} \times \frac{8 \cdot 8}{7 \cdot 9} \times \frac{9 \cdot 9}{8 \cdot 10} \times \ldots \times \frac{(n-1) \cdot (n-1)}{(n-2) \cdot n} \times \frac{n \cdot n}{(n-1) \cdot (n+1)} \\
&= 2^{n-7} \times \frac{8 \cdot n}{7 \cdot (n+1)}, n \geq 8.
\end{aligned}
$$

(b) i. $7^{333} \bmod 10$

Answer for (b.i):	7

Any number, mod 10, is just the value of its ones digit! So we are really just looking for the ones digit of 7^{333}. Next, notice (via experimentation) that ones digits of powers of 7 repeat in a cycle of length 4, which means that we can predict the ones digit of large powers of 7 using the remainder mod 4. Since $333 \equiv 1 \mod 4$, $7^{333} \equiv 7^1 \mod 10$.

ii. $32^{333} \bmod 15$

Answer for (b.ii):	2

$$\begin{aligned}
32^{333} &\equiv (2^{333})(16^{333}) \mod 15 \\
&\equiv 2^{333} \mod 15 \\
&\equiv 2^{4\cdot 83+1} \mod 15 \equiv 16^{83} \cdot 2 \mod 15 \\
&\equiv 2 \mod 15
\end{aligned}$$

(c) i. $\displaystyle\sum_{r=1}^{\infty} \left(\frac{r^2}{2^r}\right)$

Answer for (c.i):	6

$$\begin{aligned}
S(n) &= \sum_{r=1}^{\infty} \left(\frac{r^2}{2^r}\right) \\
&= \frac{1}{2} + \sum_{r=2}^{\infty} \left(\frac{r^2}{2^r}\right) \\
&= \frac{1}{2} + \sum_{r=1}^{\infty} \left(\frac{(r+1)^2}{2^{r+1}}\right) = \frac{1}{2} + \frac{1}{2}\sum_{r=1}^{\infty} \left(\frac{(r^2+2r+1)}{2^r}\right) \\
&= \frac{1}{2} + \frac{1}{2}\sum_{r=1}^{\infty} \left(\frac{r^2}{2^r}\right) + \frac{1}{2}\sum_{r=1}^{\infty} \left(\frac{2r}{2^r}\right) + \frac{1}{2}\sum_{r=1}^{\infty} \left(\frac{1}{2^r}\right) \\
&= \frac{1}{2} + \frac{1}{2}S(n) + \frac{2}{2}\sum_{r=1}^{\infty} \left(\frac{r}{2^r}\right) + \frac{1}{2} \cdot 1
\end{aligned}$$

We are getting close! Now we need an expression for $T(n) = \sum_{r=1}^{\infty} \left(\frac{2r}{2^r}\right)$:

$$\begin{aligned}
T(n) &= \frac{1}{2} + \sum_{r=2}^{\infty}\left(\frac{r}{2^r}\right) = \frac{1}{2} + \sum_{r=1}^{\infty}\left(\frac{r+1}{2^{r+1}}\right) \\
&= \frac{1}{2} + \frac{1}{2}\sum_{r=1}^{\infty}\left(\frac{r+1}{2^r}\right) \\
&= \frac{1}{2} + \frac{1}{2}\sum_{r=1}^{\infty}\left(\frac{r}{2^r}\right) + \frac{1}{2}\sum_{r=1}^{\infty}\left(\frac{1}{2^r}\right) \\
&= \frac{1}{2} + \frac{1}{2}T(n) + \frac{1}{2}\cdot 1
\end{aligned}$$

Solve for $T(n)$ to get $T(n) = 2$. And now we have:

$$\begin{aligned}
S(n) &= \frac{1}{2} + \frac{1}{2}S(n) + T(n) + \frac{1}{2}\cdot 1 \\
&= \frac{1}{2} + \frac{1}{2}S(n) + 2 + \frac{1}{2}\cdot 1 \\
&= 3 + \frac{1}{2}S(n)
\end{aligned}$$

Finally, solve for $S(n)$ to get $S(n) = 6$.

ii. $\displaystyle\sum_{r=12}^{\infty}\left(\frac{4}{5}\right)^r$

Answer for (c.ii):	$5\left(\frac{4}{5}\right)^{12}$

$$\begin{aligned}
\sum_{r=12}^{\infty}\left(\frac{4}{5}\right)^r &= \sum_{r=0}^{\infty}\left(\frac{4}{5}\right)^r - \sum_{r=0}^{11}\left(\frac{4}{5}\right)^r \\
&= \frac{1}{1-\frac{4}{5}} - \frac{1-(\frac{4}{5})^{12}}{1-\frac{4}{5}} \\
&= 5\left(\frac{4}{5}\right)^{12}
\end{aligned}$$

(d) i. $8^{(\log_2 n)/3}$

Answer for (d.i):	n

ii. $\dfrac{\log_5 4192}{\log_5 64} + \dfrac{\log_3 4192}{\log_3 64}$

Answer for (d.ii):	$\approx 2 + 2 = 4$

Note that we meant this question to ask about 4096 instead of 4192, so we're going to just give estimates of the answers, and we will be very generous in our marking. $\log_{64} 4192 \approx 2$

iii. $\dfrac{\log_3 n}{\log_{81} n}$

Answer for (d.iii):	$\log_3 81 = 4$

3. (9 points)

   (a) (2 points) Fill in the blanks:

   Theorem: For any $c \in \underline{\ \mathbb{Z}\ }$, and for any $n \in \underline{\ \mathbb{N}\ }$, $\dfrac{c^n - 1}{c - 1}$ is an integer.

   (b) (3 points) Prove the theorem from the previous part.

   This theorem is just a different way of expressing the sum of a geometric series: $\displaystyle\sum_{k=0}^{n-1} c^k = \frac{c^n - 1}{c - 1}$. Since c and n are integers, then the sum is an integer.

   (c) (4 points) Prove that $2^n \cdot 3^{2n} - 1$ is divisible by 17.

   The expression $2^n \cdot 3^{2n} - 1$ can be rewritten as $18^n - 1$. From the previous theorem, $\dfrac{18^n - 1}{17}$ is an integer. Thus, 17 divides $2^n \cdot 3^{2n} - 1$.

4. (12 points)

(a) (3 points) Complete the 4-by-4 magic square below so that every row, every column, and every diagonal have the same sum, and so that each integer from 1 through 16 is used exactly once.

14	1	**8**	**11**
15	5	**4**	**10**
2	16	9	**7**
3	**12**	**13**	**6**

(b) (3 points) Give an expression for the row sum of an n-by-n magic square.

Answer for (b):	$\frac{n(n^2+1)}{2}$

The row sum is the sum of all the values in the table, divided by n, the number of rows. Since the numbers in the table are $1 \dots n^2$, The total value of all cells is $\sum_{k=1}^{n^2} k = \frac{n^2(n^2+1)}{2}$.

(c) (3 points) Consider an arrangement of the positive integers, grouped as shown, so that the kth group has k elements: $(1), (2,3), (4,5,6), (7,8,9,10), \dots$. Find a formula for the sum of the k numbers in the kth group.

Answer for (c):	$\frac{k(k^2+1)}{2}$

(d) (3 points) Prove that your formula from the previous part is correct.

The kth group of k elements ends at value $1+2+\dots+k = \frac{k(k+1)}{2}$, and the previous group ends at value $1+2+\dots+(k-1) = \frac{k(k-1)}{2}$, so the sum of interest is:

$$
\begin{aligned}
\sum_{i=1}^{\frac{k(k+1)}{2}} i - \sum_{i=1}^{\frac{k(k-1)}{2}} i &= \frac{1}{2}\left(\frac{k(k+1)}{2}\right) \cdot \left(\frac{k(k+1)}{2} + 1\right) - \frac{1}{2}\left(\frac{k(k-1)}{2}\right) \cdot \left(\frac{k(k-1)}{2} + 1\right) \\
&= \frac{1}{8}\left(k(k+1) \cdot (k(k+1)+2) - k(k-1) \cdot (k(k-1)+2)\right) \\
&= \frac{1}{8}\left[(k^2+k) \cdot (k^2+k+2) - (k^2-k) \cdot (k^2-k+2)\right] \\
&= \frac{1}{8}\left[(k^4+k^3+2k^2+k^3+k^2+2k) - (k^4-k^3+2k^2-k^3+k^2-2k)\right] \\
&= \frac{1}{8}(4k^3+4k) \\
&= \frac{k(k^2+1)}{2}
\end{aligned}
$$

5. (12 points) Indicate for each of the following pairs of expressions $(f(n), g(n))$, whether $f(n)$ is O, Ω, or Θ of $g(n)$. Prove your answers. Note, if you choose O or Ω, you must also show that the relationship is not Θ.

   (a) $f(n) = n \log n$, and $g(n) = n^2 \log n - n \log(n^2 + 1)$.

Answer for (a):	$f(n) = O(g(n)$

We will show that if $n \geq 4$, $n \log n \leq n^2 \log n - n \log(n^2 + 1)$. Note that this inequality holds, if and only if $n \log n + n \log(n^2 + 1) \leq n^2 \log n$.

$$\begin{aligned} n \log n + n \log(n^2 + 1) &\leq n \log n + n \log 2n^2 \\ &\leq n \log n + n \log 2 + 2n \log n \\ &\leq n \log n + n \log n + 2n \log n \\ &= 4n \log n \\ &\leq n^2 \log n \end{aligned}$$

where the last inequality holds since $n \geq 4$.
Next we show that $f(n) \notin \Omega(g(n))$.
Consider an arbitrary c and n_0. We will show that there is an n so that $n \log n + cn \log(n^2 + 1) \leq cn^2 \log n$.
Let $n = \max\{n_0, \frac{1+3c}{c}\}$. Then,

$$\begin{aligned} & 1 + 3c &&\leq cn \\ \iff\quad & (1 + 3c)n \log n &&< cn^2 \log n \\ \implies\quad & n \log n + 2cn \log n + cn \log n &&\leq cn^2 \log n \\ \implies\quad & n \log n + 2cn \log n + cn \log 2 &&\leq cn^2 \log n \\ \implies\quad & n \log n + cn \log(n^2 + 1) &&\leq cn^2 \log n \end{aligned}$$

   (b) $f(n) = n^3$, and $g(n) = \frac{1}{2}n^2 - 4n - 37$.

Answer for (b):	$f(n) \in \Omega(g(n))$

Let $c = n_0 = 1$, and consider an arbitrary $n \geq 1$. Rather trivially, $\dfrac{1}{2}n^2 - 4n - 37 \leq n^2 \leq n^3$, so $f(n) \in \Omega(g(n))$.
Now consider arbitrary c and n_0, and let $n = \max\{n_0, \frac{c}{2}\}$. Then,

$$
\begin{aligned}
& n && \geq \tfrac{c}{2} \\
\implies\quad & n^3 && \geq \tfrac{c}{2}n^2 \\
\implies\quad & n^3 && \geq \tfrac{c}{2}n^2 - c4n - 37c
\end{aligned}
$$

6. (16 points) For each C++ function below, give the tightest asymptotic upper bound that you can determine for the function's **runtime**, in terms of the input parameter. Where prompted, also compute the return value of the function. For all function calls, assume that the input parameter $n \geq 2$.

   (a)

```cpp
int honeydoughnut(int n) {
   int s = 0;
   for (int i = 1; i < n * n * n; i = i * 2) {
         cout << "iteration: " << i << endl;
         s++;
   }
   return s;
}
```

Return value for (a):	$3 \lg n$

Running time of (a):	$O(\log n)$

   (b)

```cpp
int vanillawafer(int n) {
   int s = n / 2;
   for (int i = 0; i < 573000; i++)
         s++;
   return s;
}
```

Return value for (b):	$\lfloor \frac{n}{2} \rfloor + 573000$

Running time of (b):	$O(1)$

(c)

```
int strawberrysmoothie(int n) {
   int s = 0;
   for (int i = 0; i < n; i++) {
         if (i % 2 == 0)
            s += i;
   }
   return s;
}
```

Return value for (c):	$\lfloor \frac{n-1}{2} \rfloor \cdot \left(\lfloor \frac{n-1}{2} \rfloor + 1 \right)$

Running time of (c):	$O(n)$

(d)

```
int cherrycreamtrifle(int n) {
   int j = 0;

   for (int k = 0; k <= n; k++)
         j = j + strawberrysmoothie(k);

   for (int m = 1; m < j; m++)
         cout << "I am having so much fun with asymptotics!" << endl;

   return n + (n % 41);
}
```

Running time of (d):	$O(n^3)$

(e)

```
int rockyroadcaramelchocolatechunkcookieicecream(int n) {

   return honeydoughnut(cherrycreamtrifle(n));
}
```

Running time of (e):	$O(n^3)$

Blank sheet for extra work.