Algorithms

Grad Refresher Course 2011
University of British Columbia

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About this talk

- For those incoming grad students who
  - Do not have a CS background, or
  - Have a CS background from a long time ago
- Discuss some fundamental concepts from algorithms and CS theory
- Ease the transition into any grad-level CS course
- Based on the 2009 version by Brad Bingham
- Some slides used from MIT OpenCourseWare

UBC CS Theory Courses (UGrad)

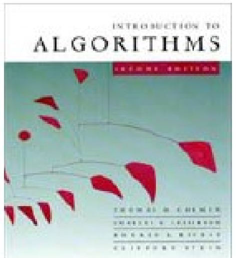
- CPSC 320: Intermediate Algorithm Design and Analysis
  - Required for CS undergrads
  - Offered in term 1 (Belleville) and term 2 (Meyer)
- CPSC 421: Intro to Theory of Computing
  - Offered in term 1 (Friedman)
- CSPC 420: Advanced Alg. Design & Analysis
  - Offered in term 2 (Kirkpatrick)

Outline

- Asymptotic Notation and Analysis
- Graphs and algorithms
- NP-Completeness & undecidability
- Resources to Learn More

Pseudocode

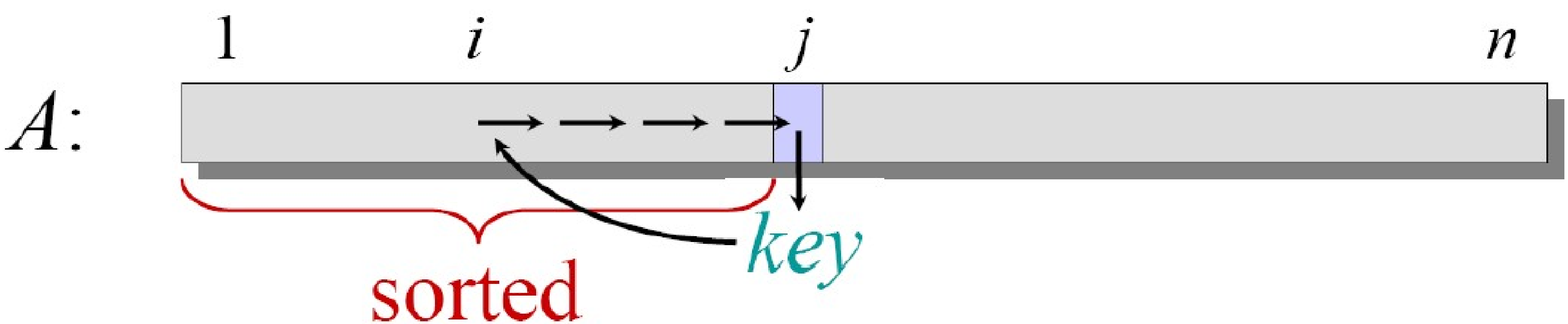
- How do we analyze algorithms? Start with a pseudocode description!
- Specifies an algorithm mathematically
- Independent of hardware details, programming languages, etc.
- Reason about ***scalability*** in a mathematical way



Insertion sort

“pseudocode”

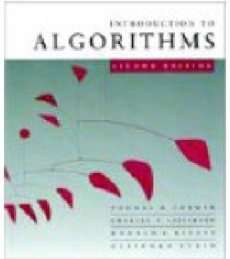
```
INSERTION-SORT (A, n)      ▷ A[1 . . n]
  for j ← 2 to n
       do key ← A[ j]
          i ← j – 1
          while i > 0 and A[i] > key
               do A[i+1] ← A[i]
                  i ← i – 1
          A[i+1] = key
```



Example of insertion sort

8 | 2 4 9 3 6

sorted | unsorted



Example of insertion sort

8 | 2 4 9 3 6

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Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6

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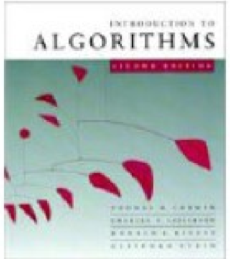
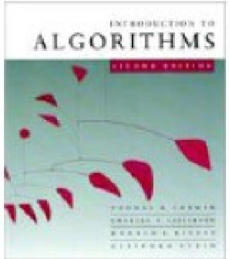


Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6

sorted unsorted



Example of insertion sort

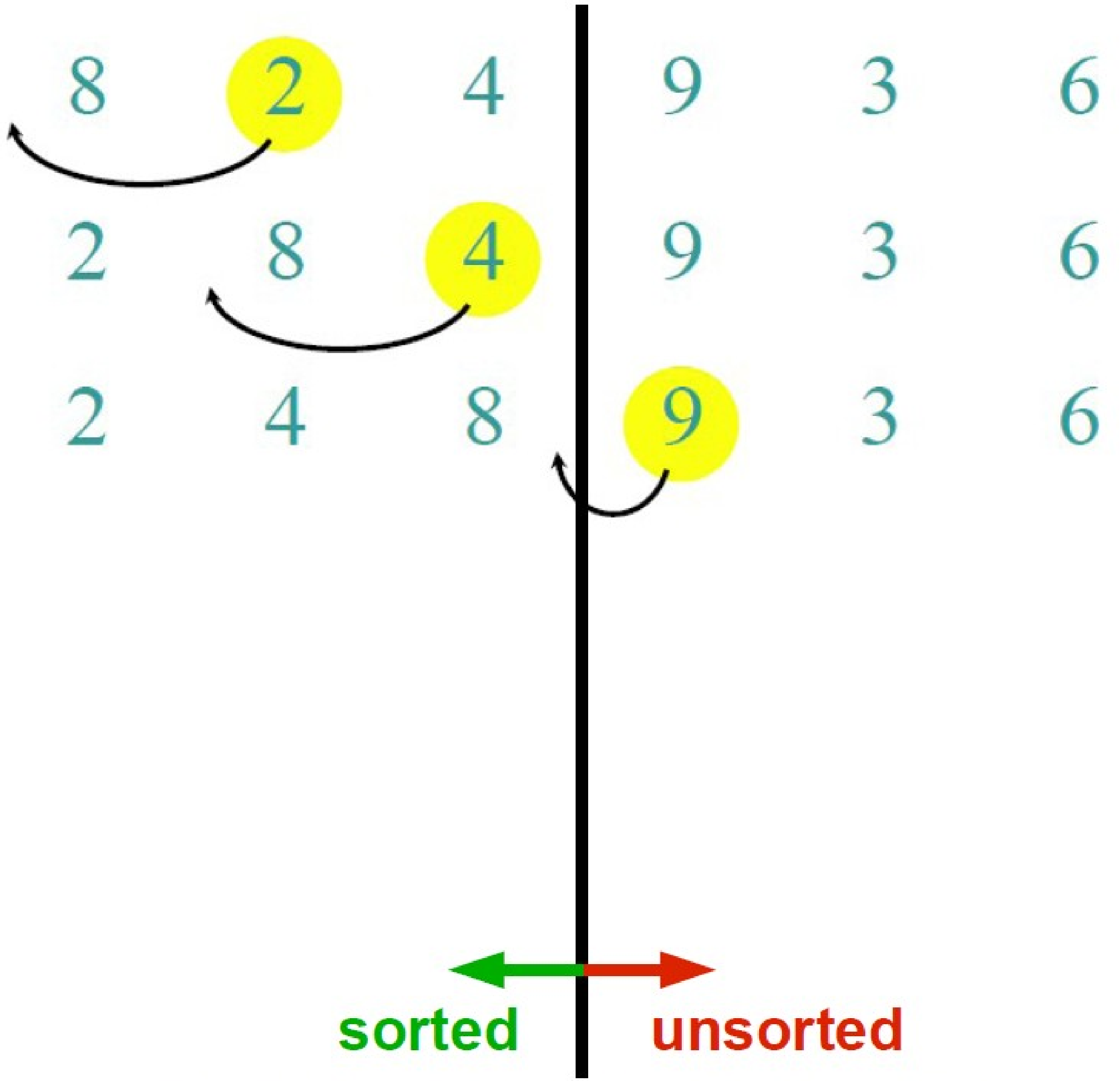
8 2 4 | 9 3 6

2 8 4 | 9 3 6

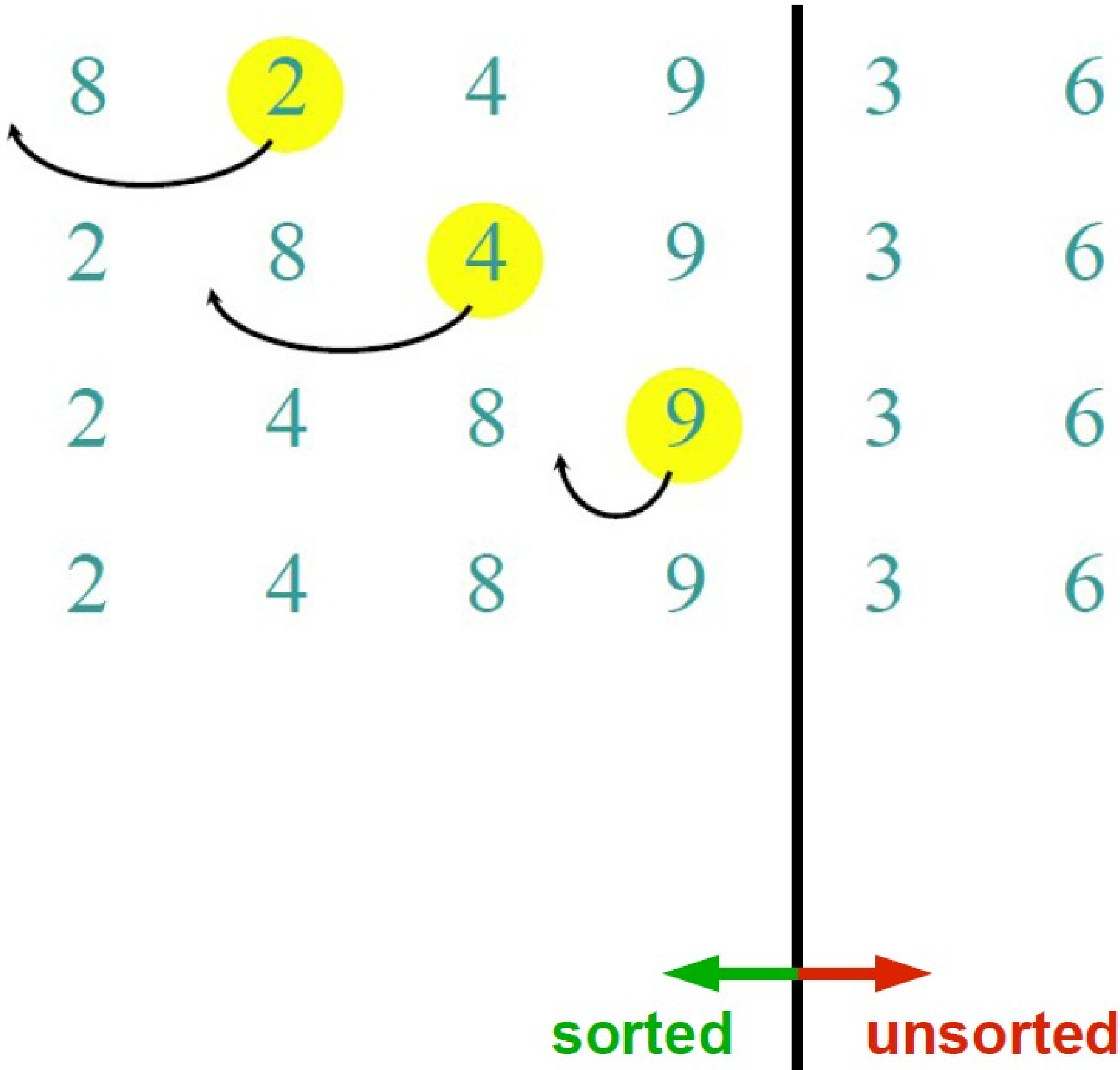
2 4 8 | 9 3 6

sorted | unsorted

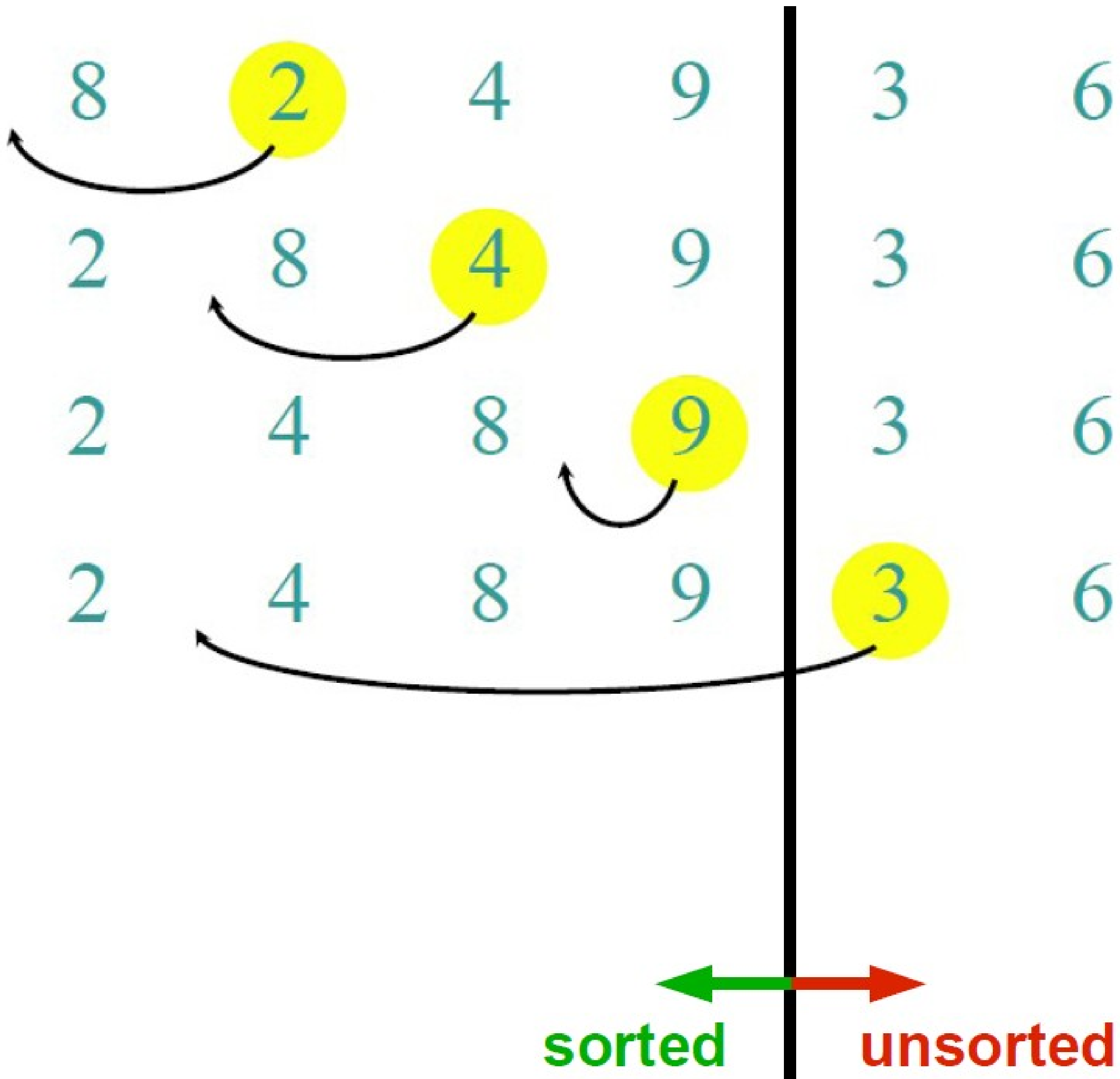
Example of insertion sort



Example of insertion sort



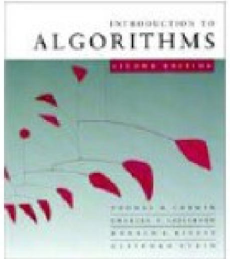
Example of insertion sort



Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6
2	4	8	9	3	6
2	3	4	8	9	6

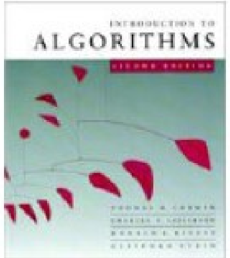
sorted unsorted



Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6
2	4	8	9	3	6
2	3	4	8	9	6

sorted unsorted



Example of insertion sort

8 2 4 9 3 6

2 8 4 9 3 6

2 4 8 9 3 6

2 4 8 9 3 6

2 3 4 8 9 6

2 3 4 6 8 9 *done*

sorted | unsorted

How to Analyze?

- Count operations in the Random Access Machine (RAM) model:
    - Single processor
    - Infinite memory, constant time reads/writes
    - “Reasonable” instruction set

  www8.cs.umu.se/kurser/TDBAfl/VT06/algorithms/BOOK/BOOK/NODE12.HTM

- Asymptotically (Big-O)
- Scenarios: worst case, average case
- What to analyze: time complexity, space complexity

Big-O Notation

O(f(n)) is a set of functions

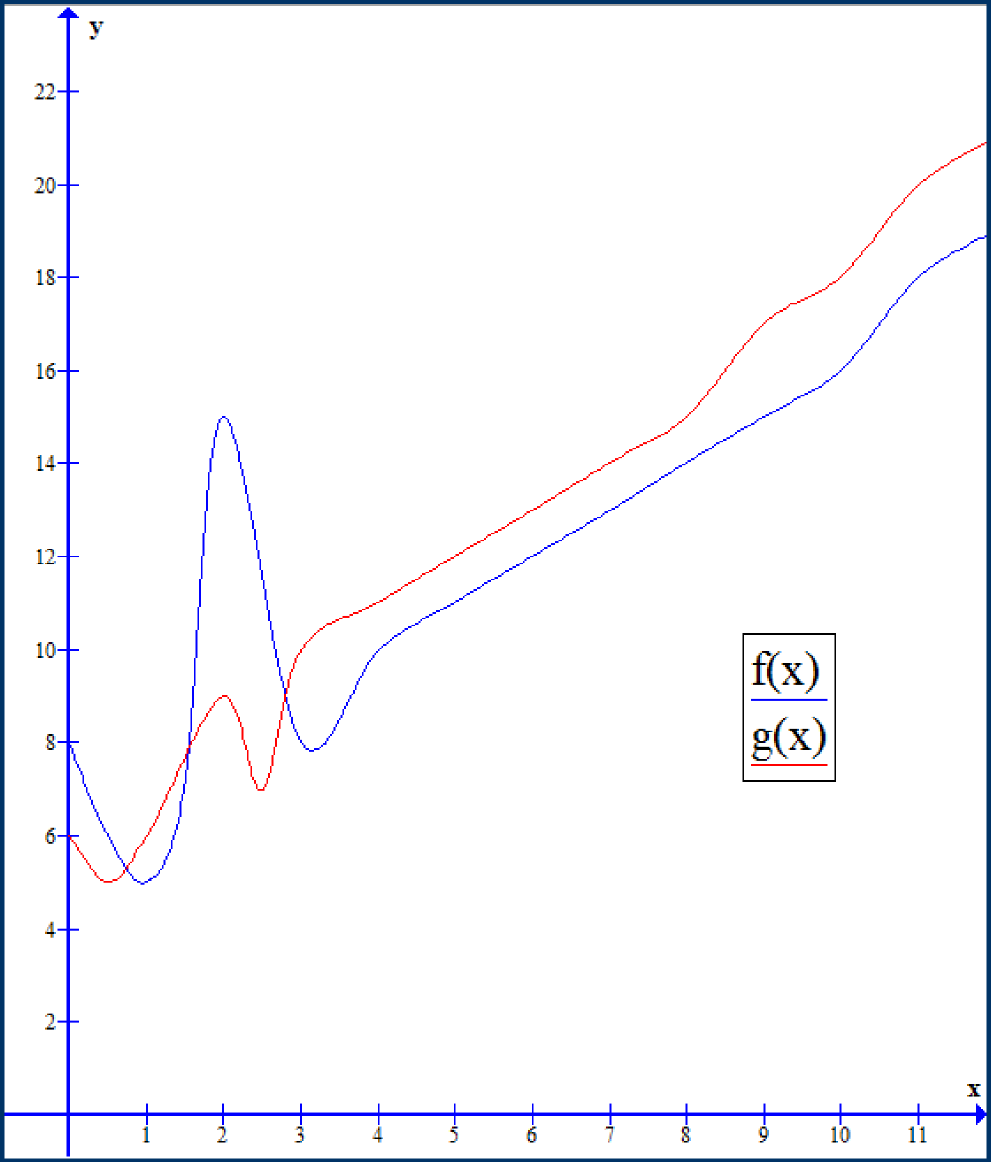
$$g(n) \in O(f(n)) \Leftrightarrow \text{there exist constants } c, n_0 \text{ such that}$$

$$g(n) \leqslant c \cdot f(n) \text{ for all } n \geqslant n_0$$

- In words:

  “for sufficiently large inputs, the function g(n) is dominated by a scaled f(n)”
- An upper bound, in an asymptotic sense
- Usually, g(n) is a complicated expression and f(n) is simple

Big-O: Illustration



Big-O Example

Show that $\frac{1}{2}n^2 + 3\mathrm{n}$ is $O(n^2)$

i.e., find constants *c* and n_0 where

$$\frac{1}{2}n^2 + 3\mathrm{n} \leqslant c \cdot n^2, \forall n \geqslant n_0$$

Solution: choose c=1, solve for n:

$$\frac{1}{2}n^2 + 3\mathrm{n} \leqslant n^2 \Rightarrow 6 \leqslant n \Rightarrow n_0 = 6$$

In general, ignore constants and drop lower order terms. For example:

$2\mathrm{n}^4 + 6\mathrm{n}^3 + 100\mathrm{n} - 27$ is $O(n^4)$

Big-Omega, Big-Theta

$$g(n) \in \Omega(f(n)) \Leftrightarrow \textit{there exist constants } c, n_0 \textit{ such that}$$

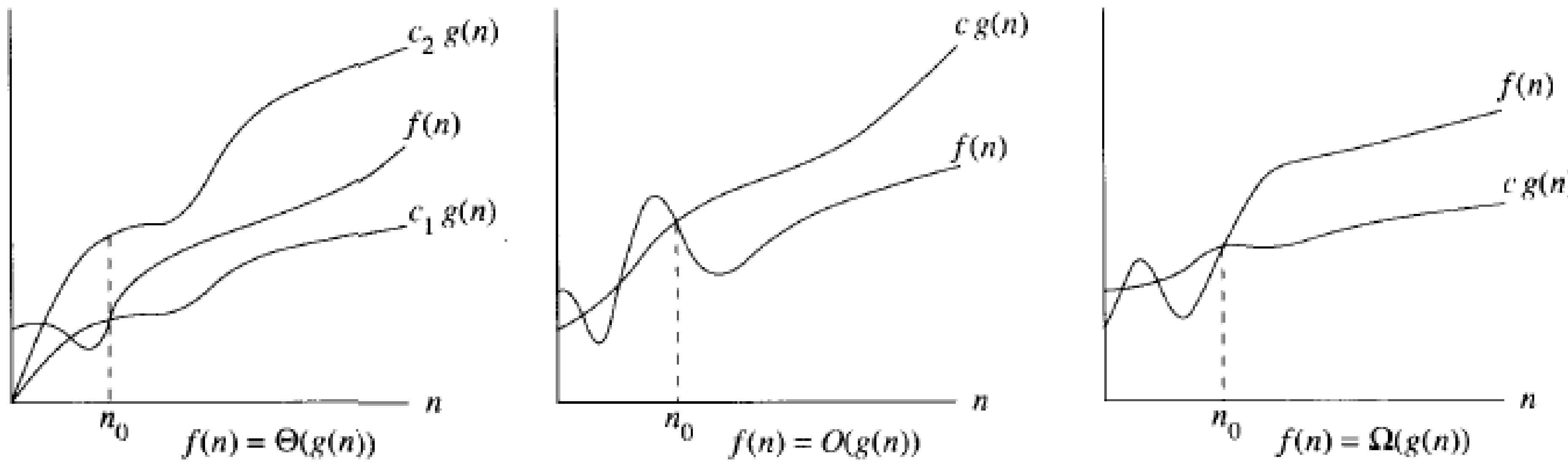
$$g(n) \geqslant c \cdot f(n) \textit{ for all } n \geqslant n_0$$

- In words:

  "for sufficiently large inputs, the function g(n) dominates a scaled f(n)"
- A lower bound, in an asymptotic sense

$g(n) \in \Theta(f(n))$ if $g(n) \in O(f(n))$ and $g(n) \in \Omega(f(n))$

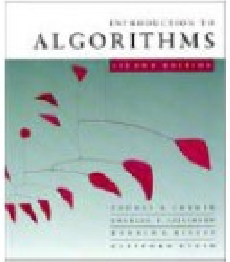
- A "tight" asymptotic bound



$c_2\, g(n)$
$f(n)$
$c_1\, g(n)$
n
n_0
$f(n) = \Theta(g(n))$
$c\, g(n)$
$f(n)$
n
n_0
$f(n) = O(g(n))$
$f(n)$
$c\, g(n)$
n
n_0
$f(n) = \Omega(g(n))$

Complexity “Food Chain”

Name	Expression
Constant	$O(1)$
Logarithmic	$O(\log(n))$
Linear	$O(n)$
Linearithmic	$O(n \log(n))$
Quadratic	$O(n^2)$
Polynomial	$O(n^p)$
Exponential	$O(2^n)$



Insertion sort

Count the number of times these lines execute

for-loop runs for n-1 iterations

while-loop runs at most j-1 iterations (on worst-case input)

```
INSERTION-SORT (A, n)        ▷ A[1 . . n]
  for j ← 2 to n
       do key ← A[j]
          i ← j – 1
          while i > 0 and A[i] > key
               do A[i+1] ← A[i]
                  i ← i – 1
          A[i+1] = key
```

- Runtime is $\sum_{j=2}^{n}(j-1) = n(n-1)/2 - 1 \in \Theta(n^2)$
- In general, all sorting algorithms* are in $\Omega(n \log(n))$

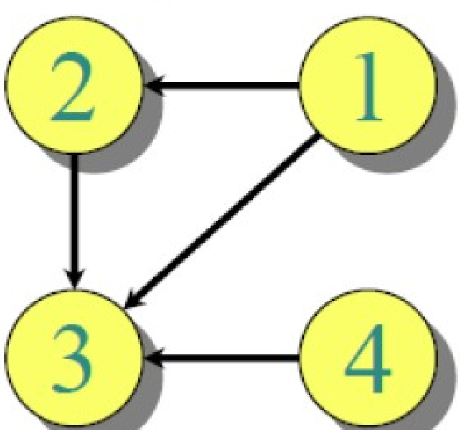
*(in the comparison model)

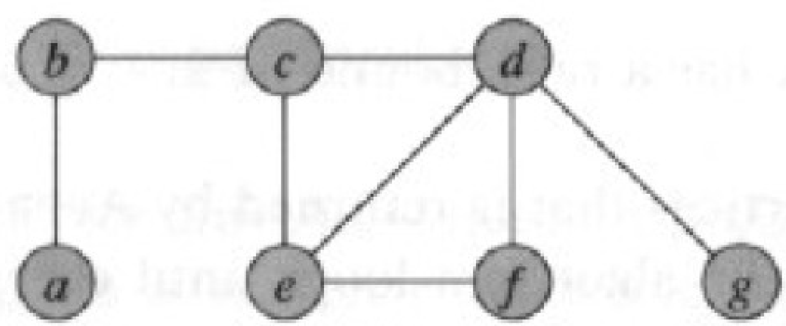
Graphs

Definition. A ***directed graph (digraph)*** $G = (V, E)$ is an ordered pair consisting of

- a set V of ***vertices*** (singular: ***vertex***),
- a set $E \subseteq V \times V$ of ***edges***.

In an ***undirected graph*** $G = (V, E)$, the edge set E consists of *unordered* pairs of vertices.



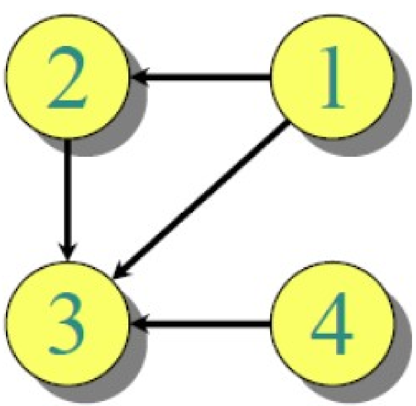


(a)

Adjacency-matrix representation

The ***adjacency matrix*** of a graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, is the matrix $A[1 \,.\,.\, n, 1 \,.\,.\, n]$ given by

$$A[i, j] = \begin{cases} 1 & \text{if } (i, j) \in \mathrm{E}, \\ 0 & \text{if } (i, j) \notin \mathrm{E}. \end{cases}$$



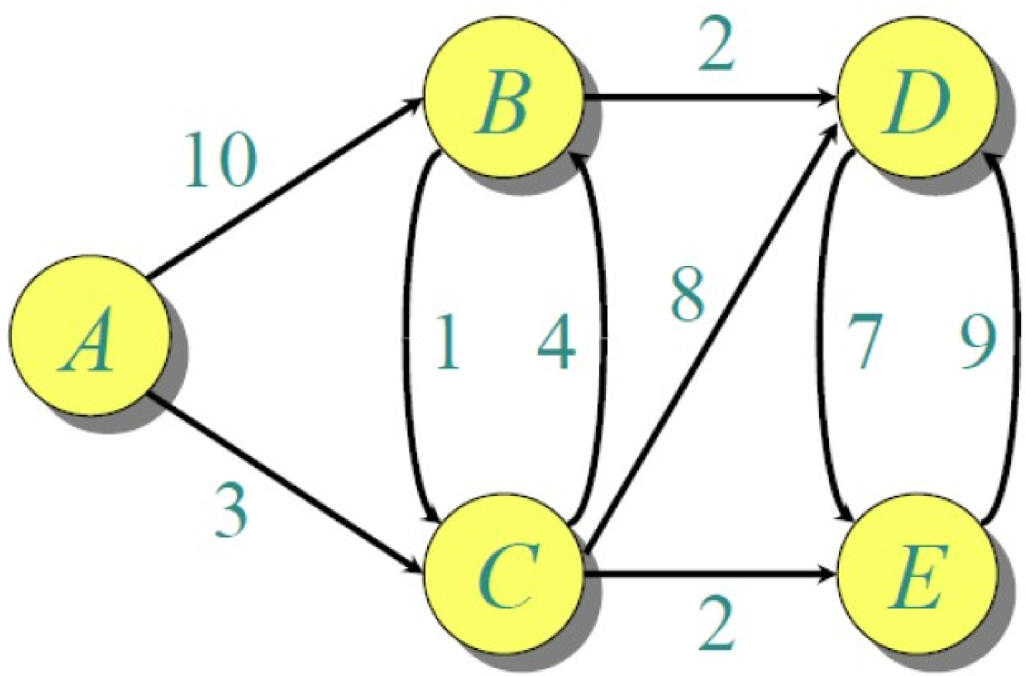
A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

$\Theta(V^2)$ storage $\Rightarrow$ ***dense*** representation.

Single-source shortest paths

Problem. From a given source vertex $s \in V$, find the shortest-path weights $\delta(s, v)$ for all $v \in V$.

If all edge weights $w(u, v)$ are *nonnegative*, all shortest-path weights must exist.



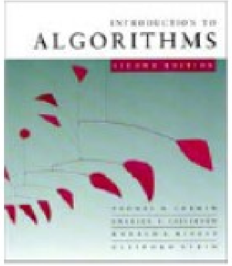
Single-source shortest paths

Problem. From a given source vertex $s \in V$, find the shortest-path weights $\delta(s, v)$ for all $v \in V$.

If all edge weights $w(u, v)$ are *nonnegative*, all shortest-path weights must exist.

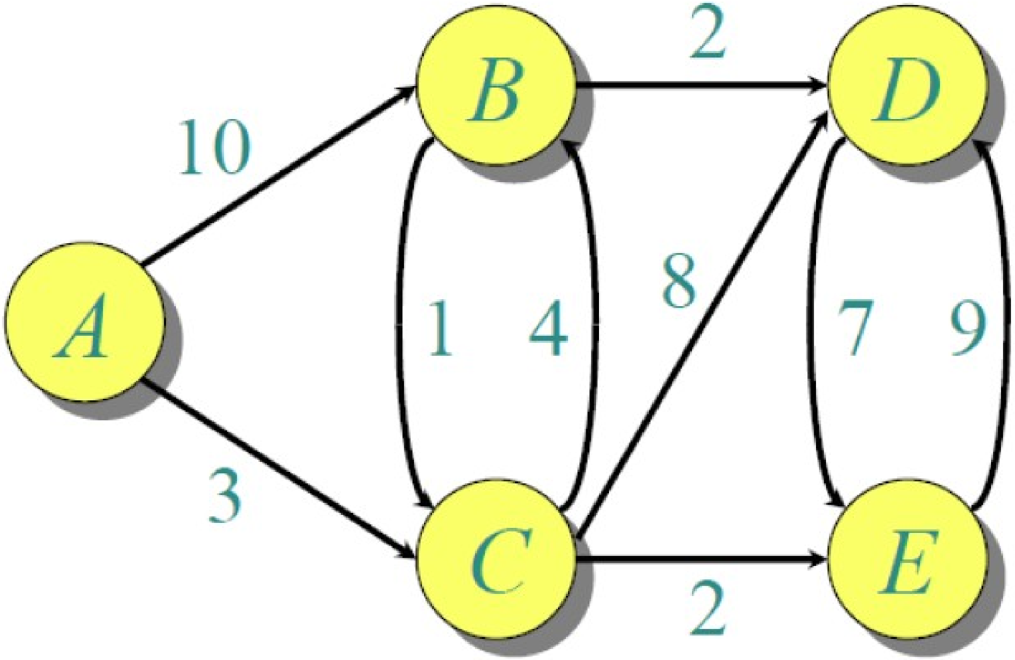
IDEA: Greedy.

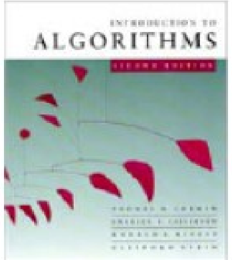
1. Maintain a set S of vertices whose shortest-path distances from s are known.
2. At each step add to S the vertex $v \in V - S$ **= *Q*** whose distance estimate from s is minimal.
3. Update the distance estimates of vertices adjacent to v.



Example of Dijkstra's algorithm

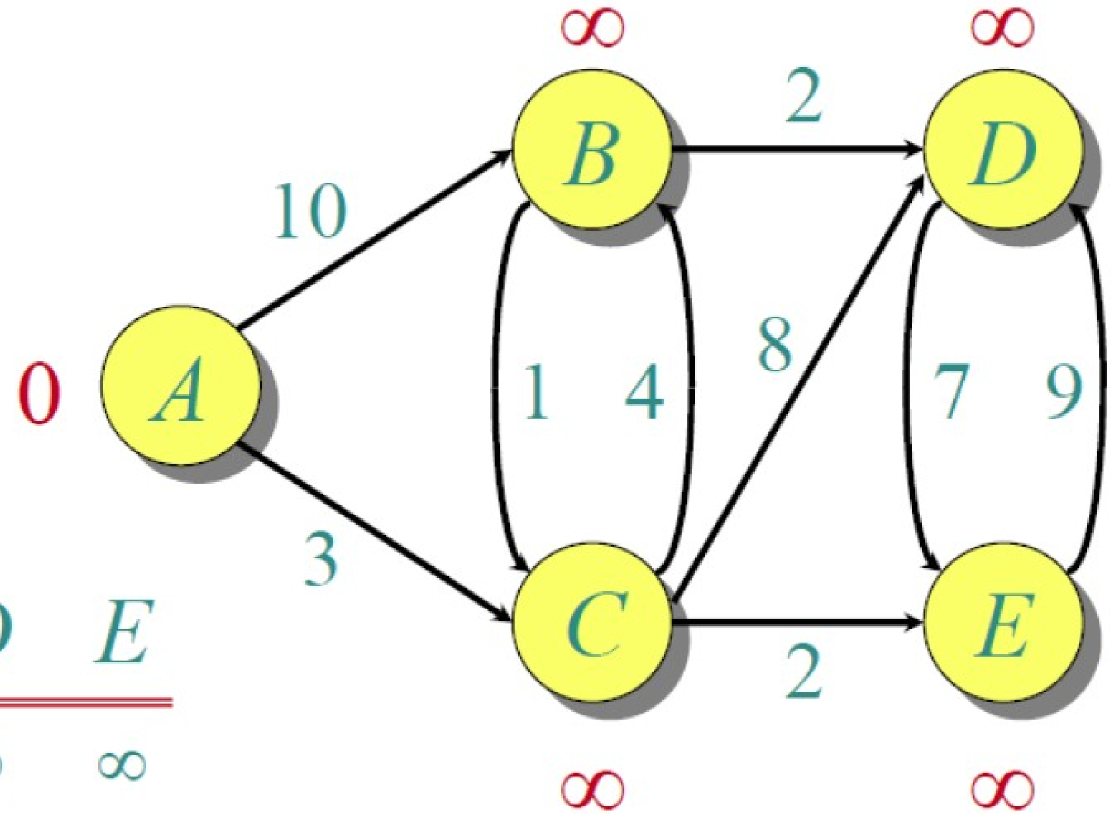
Graph with nonnegative edge weights:





Example of Dijkstra's algorithm

Initialize:

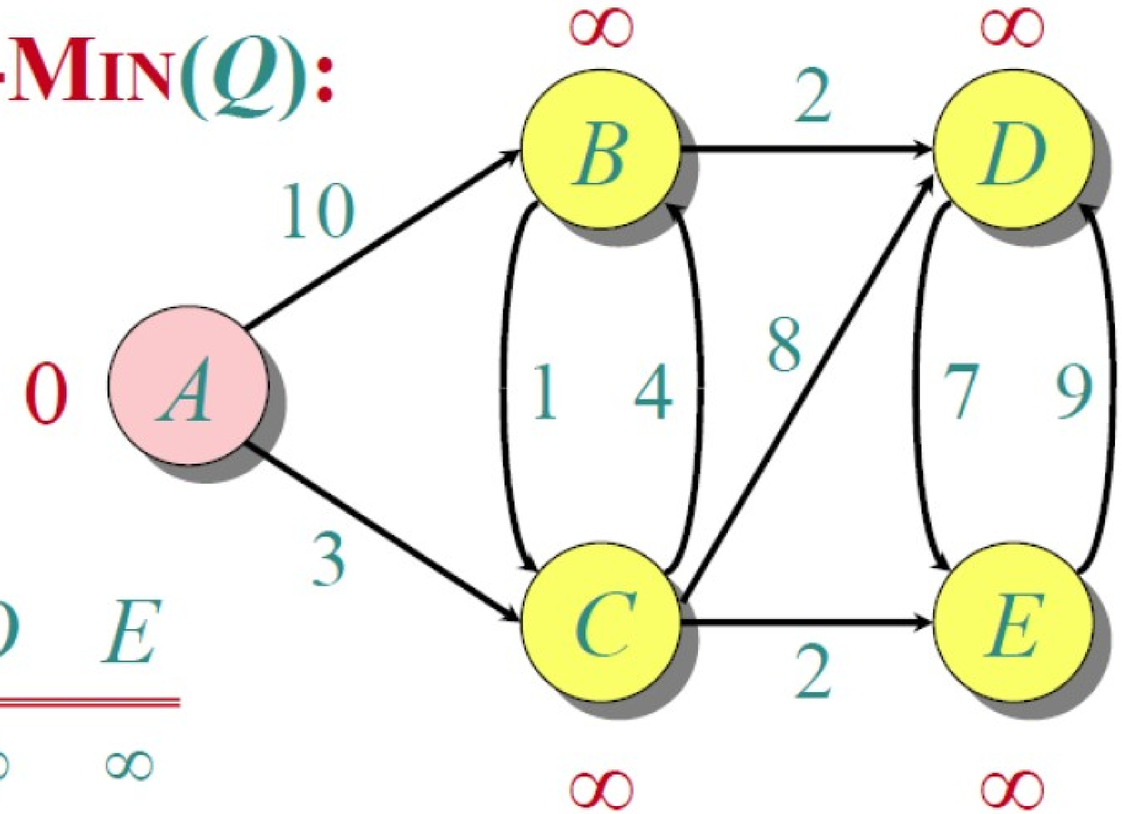


Q:	A	B	C	D	E
	0	∞	∞	∞	∞

S: {}

Example of Dijkstra's algorithm

"A" ← EXTRACT-MIN(Q):

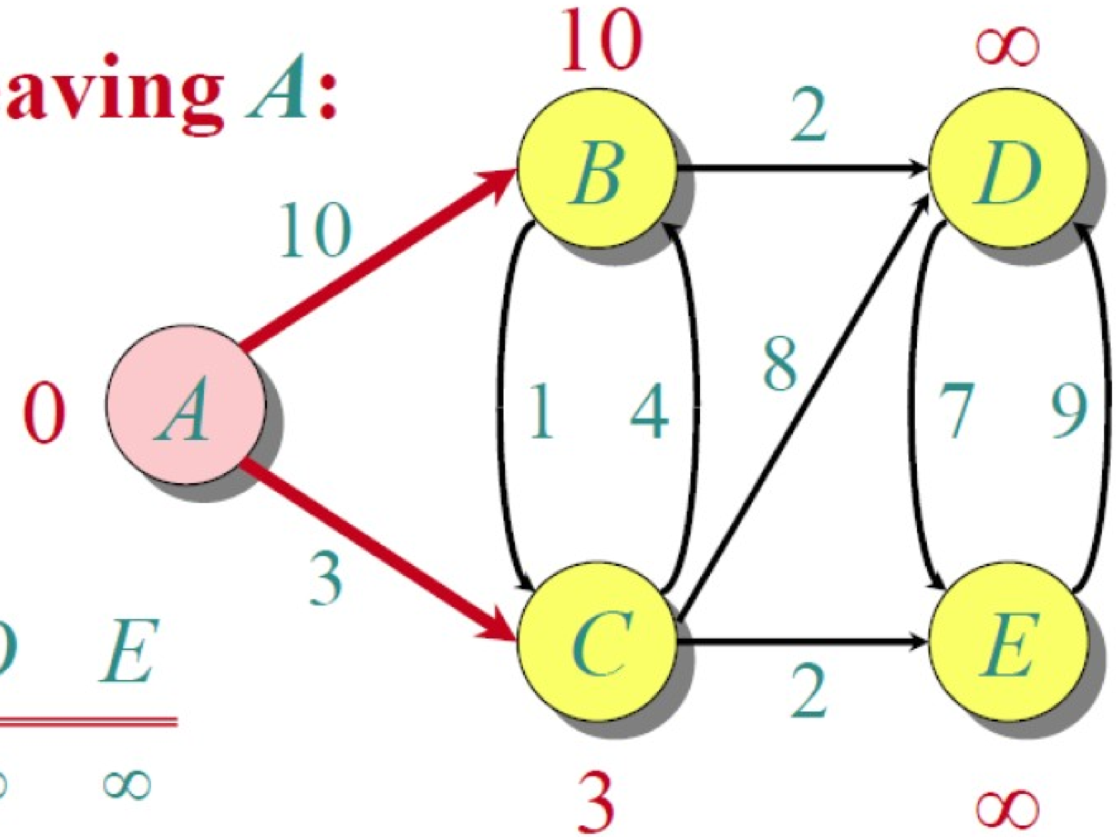


Q:	A	B	C	D	E
	0	∞	∞	∞	∞

S: { A }

Example of Dijkstra's algorithm

Relax all edges leaving *A*:

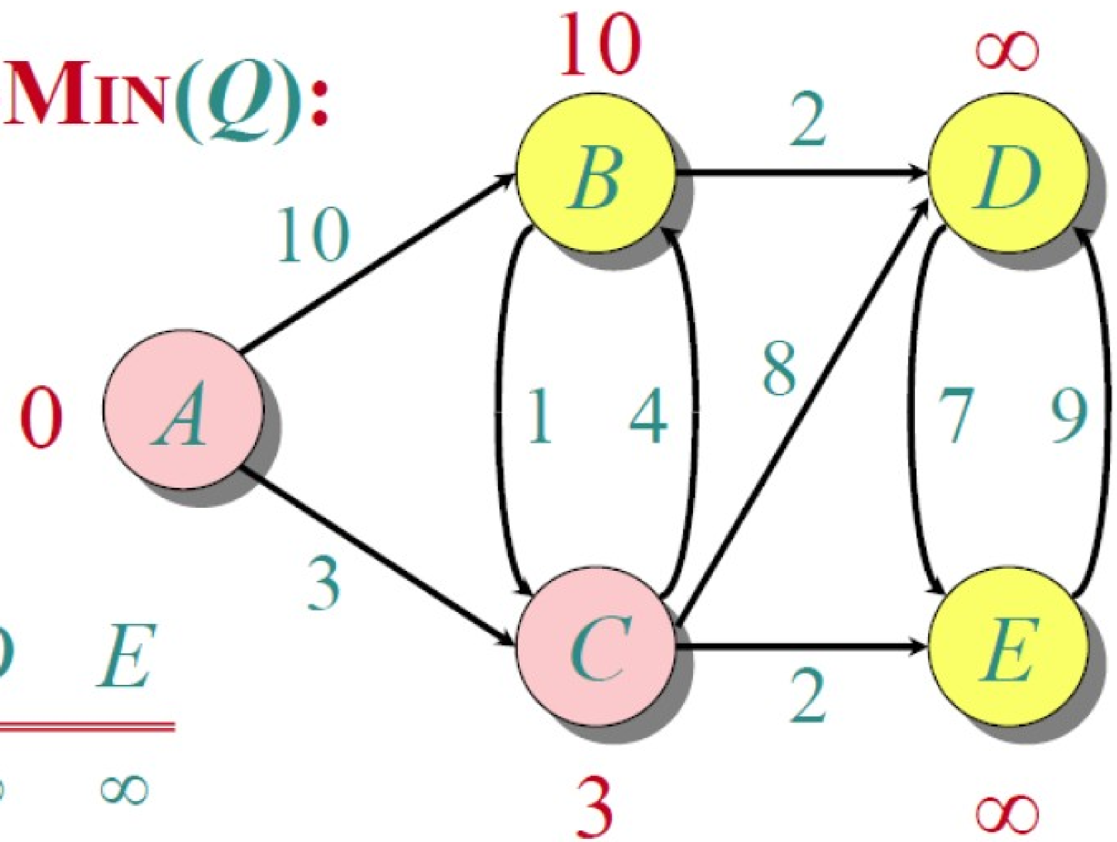


Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞

S: { A }

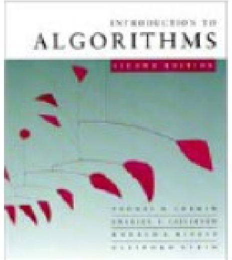
Example of Dijkstra's algorithm

"C" $\leftarrow$ EXTRACT-MIN(Q):



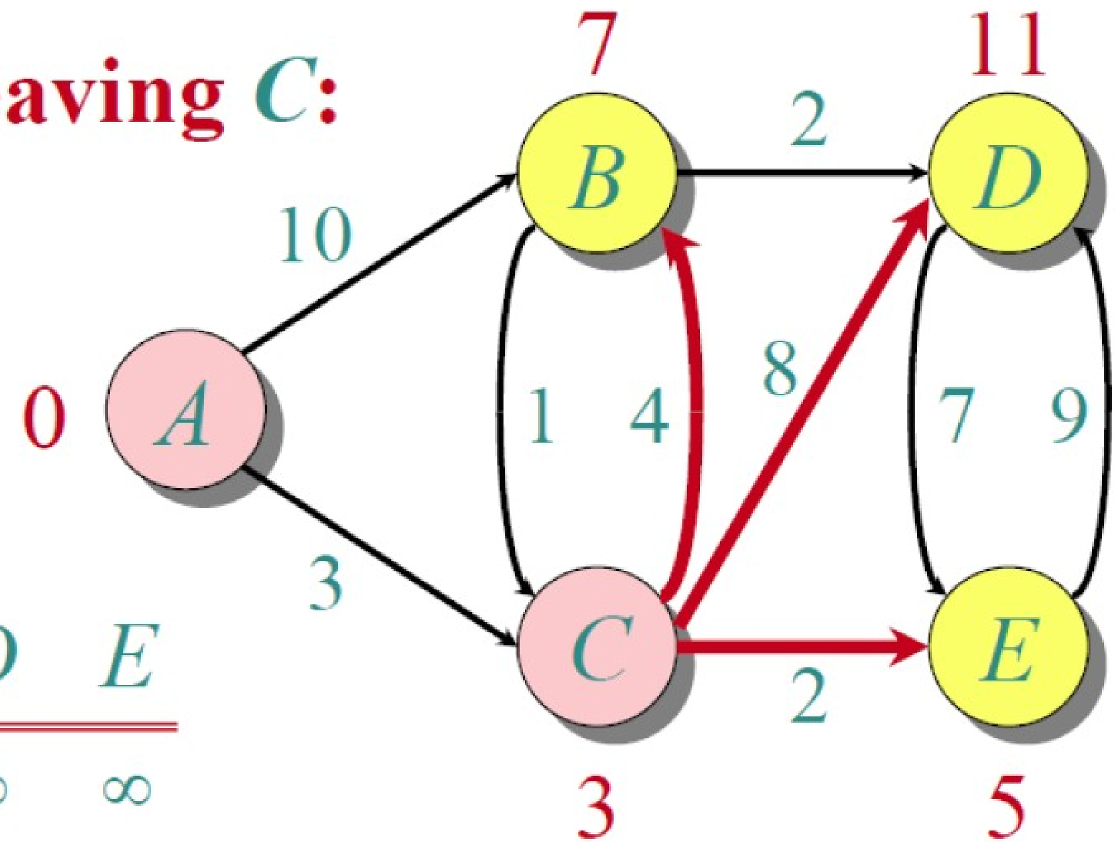
Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞

S: { A, C }



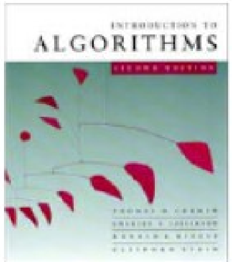
Example of Dijkstra's algorithm

Relax all edges leaving *C*:



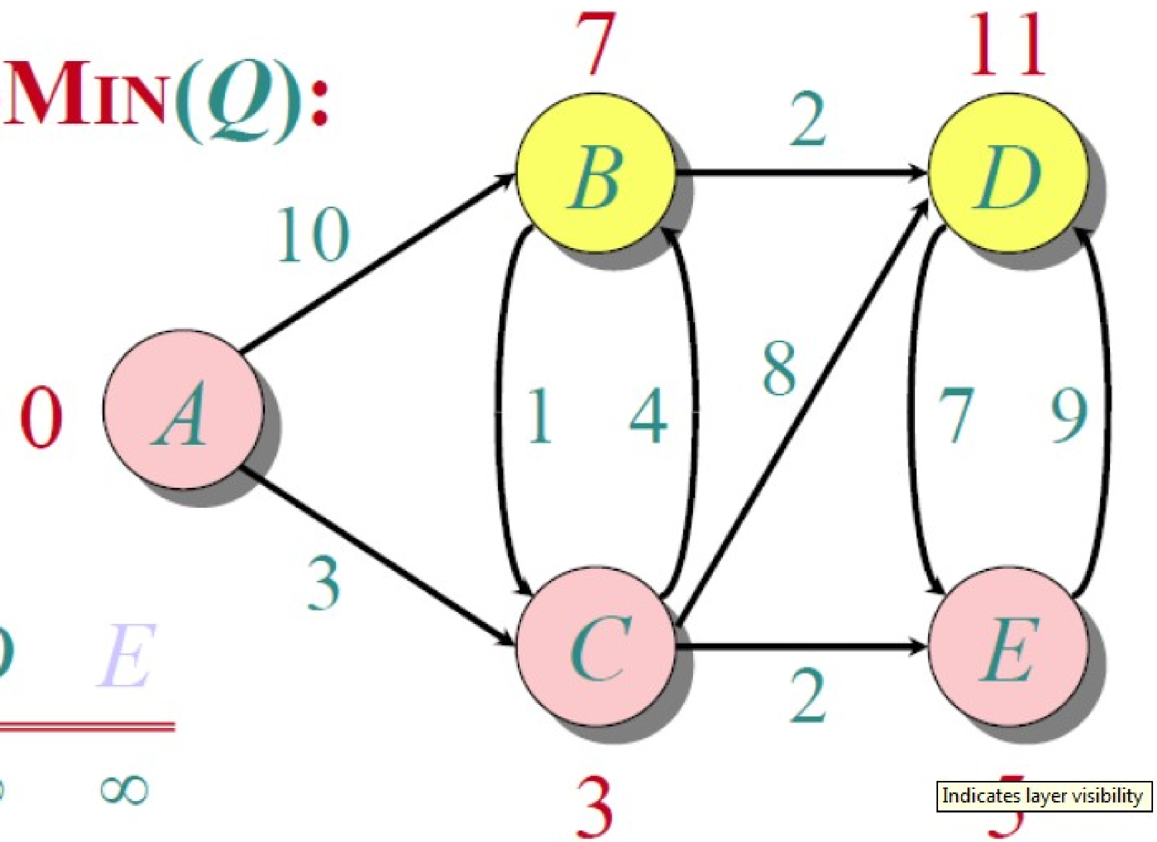
Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

S: { *A*, *C* }



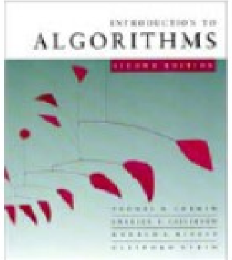
Example of Dijkstra's algorithm

"E" ← EXTRACT-MIN(Q):



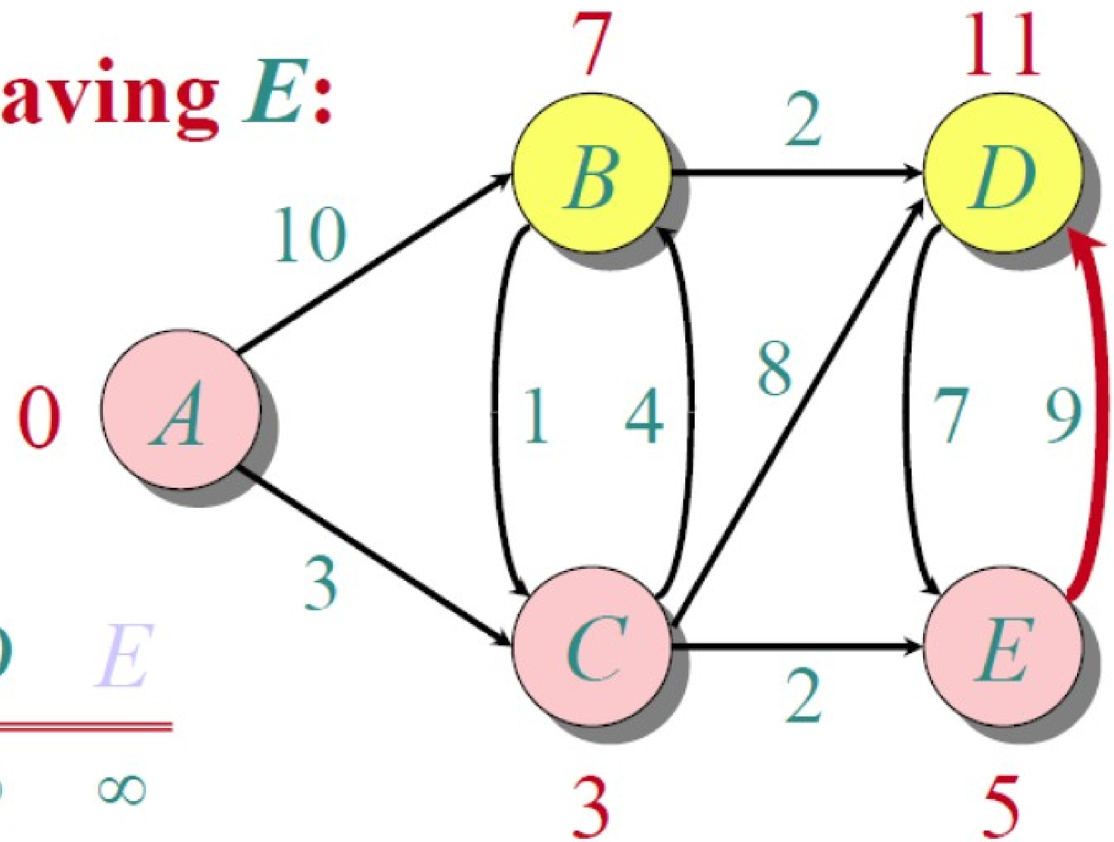
Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

S: { A, C, E }



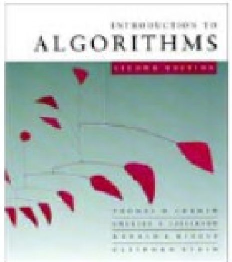
Example of Dijkstra's algorithm

Relax all edges leaving *E*:



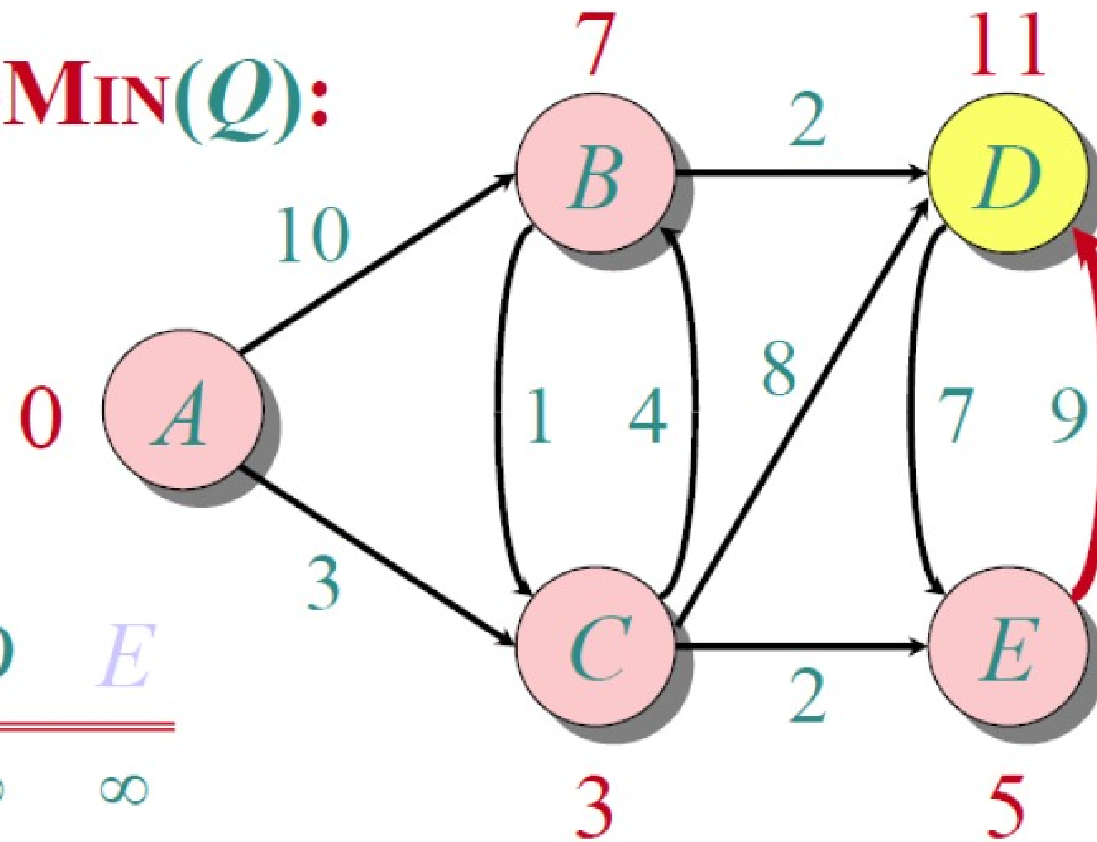
Q:	A	B	C	D	E
	$\boxed{0}$	∞	∞	∞	∞
		10	$\boxed{3}$	∞	∞
		7		11	$\boxed{5}$
		7		11	

S: { *A*, *C*, *E* }



Example of Dijkstra's algorithm

"B" $\leftarrow$ EXTRACT-MIN(Q):

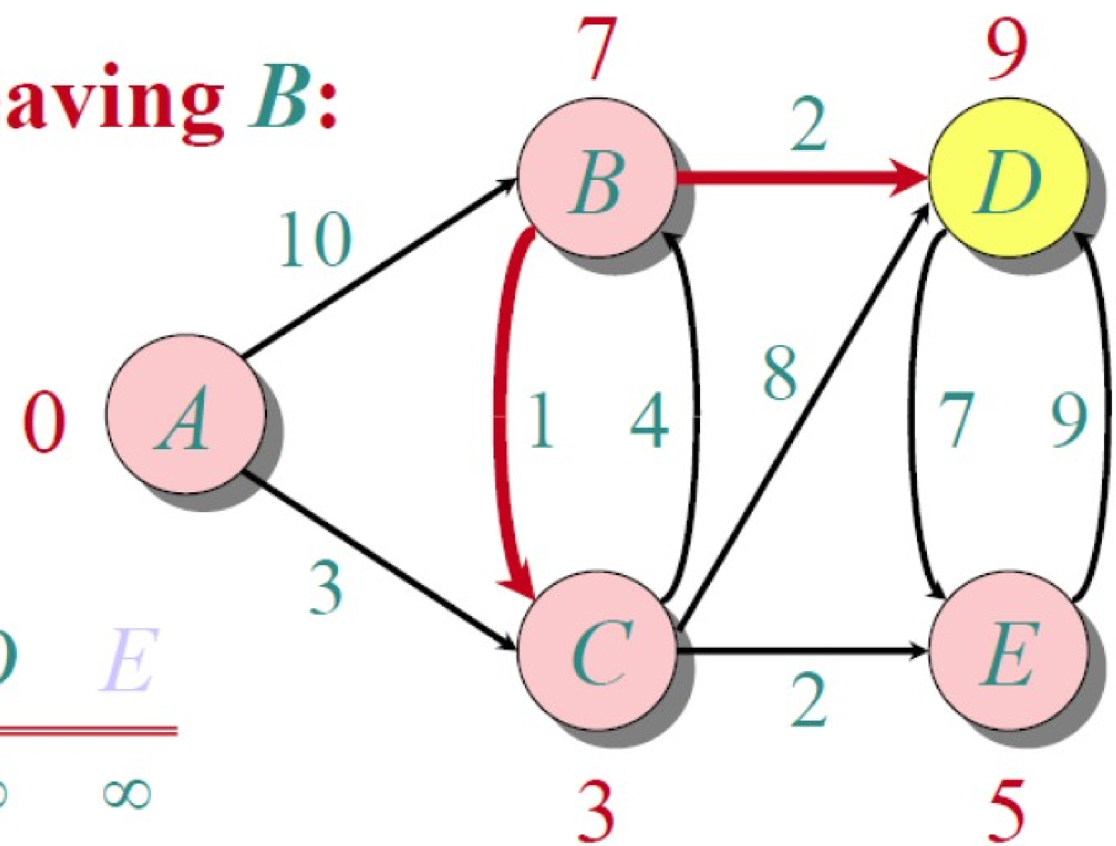


Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { A, C, E, B }

Example of Dijkstra's algorithm

Relax all edges leaving *B*:

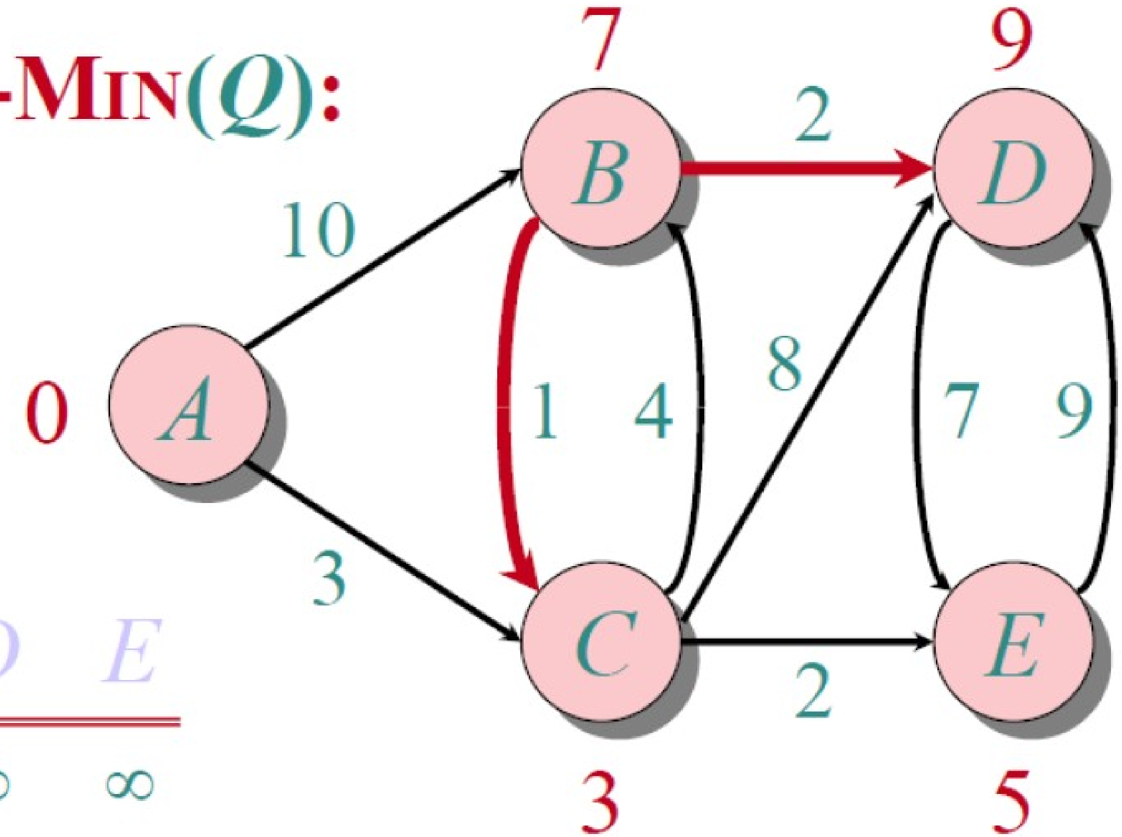


Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { A, C, E, B }

Example of Dijkstra's algorithm

"D" $\leftarrow$ EXTRACT-MIN(Q):



Q:	A	B	C	D	E
	$\boxed{0}$	∞	∞	∞	∞
		10	$\boxed{3}$	∞	∞
		7		11	$\boxed{5}$
		$\boxed{7}$		11	
				$\boxed{9}$	

S: { A, C, E, B, D }

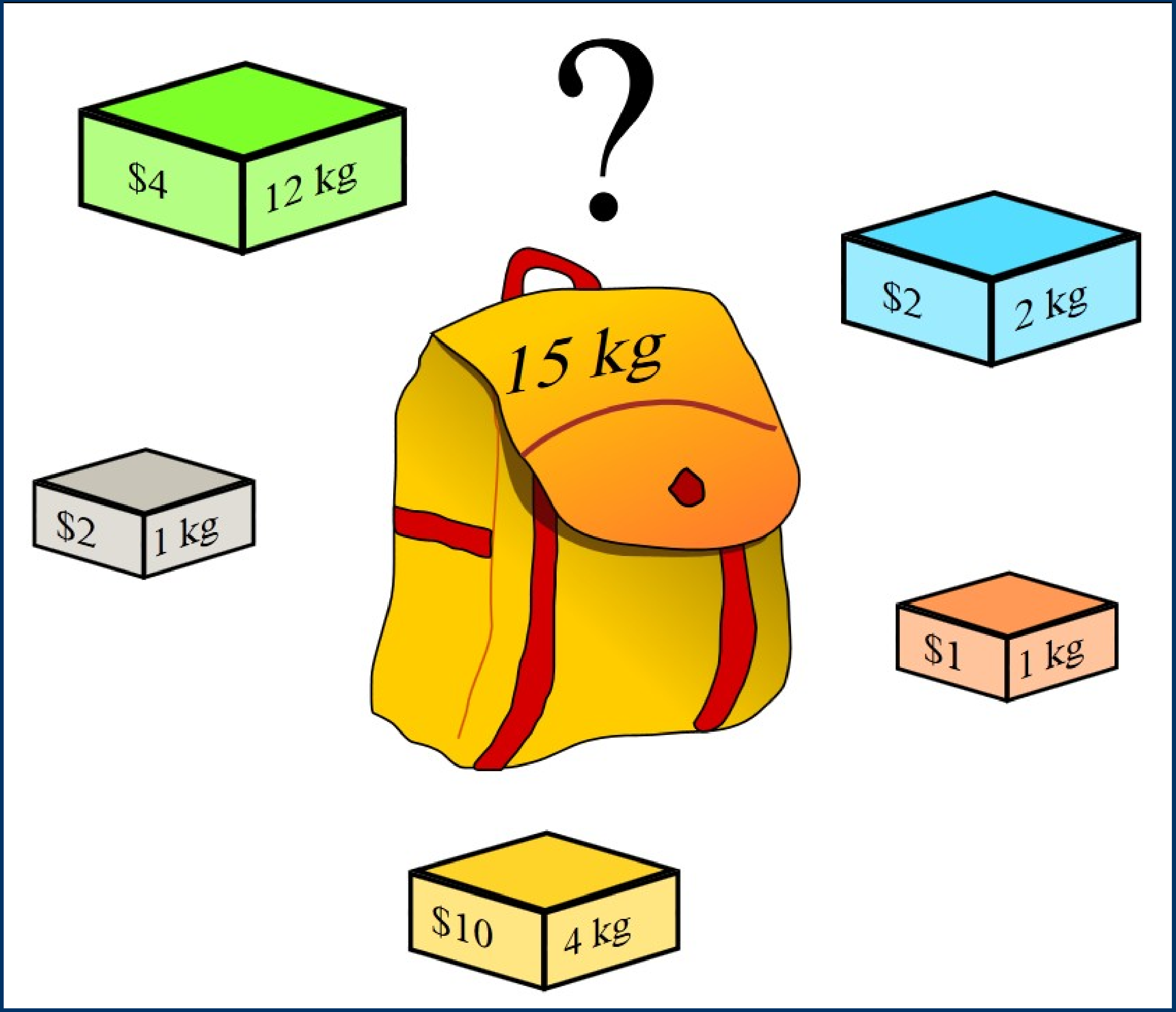
Traveling Salesman Problem (TSP)

- The run time of Dijkstra's algorithm in $O(n^2)$ with naïve data structures. This is polynomial, so we say the shortest-path problem can be solved in "polynomial time".

- What about the following (similar) problem?
- Given a directed graph with edge weights, find a path that
    1) Visits all vertices, and
    2) Minimizes the path weight (sum of edges)

- There is no known algorithm for solving this in polynomial time. Why? **TSP is NP-complete**.

Knapsack Problem



NP-Completeness

- NP-complete problems are a class of problems for which there is no known algorithm that run in $O(n^k)$ time, for *any* constant k
- Equivalently, all known algorithms for solving NP-complete problems are likely to be unacceptably slow
- If such an algorithm is found, or proven to not to exist, this solves the famous “P=NP?” question (such a proof is worth $1 Million USD)
- NP-complete problems are *verifiable* in polynomial time
- NP-hard: problems that are “at least as hard as NP-complete”

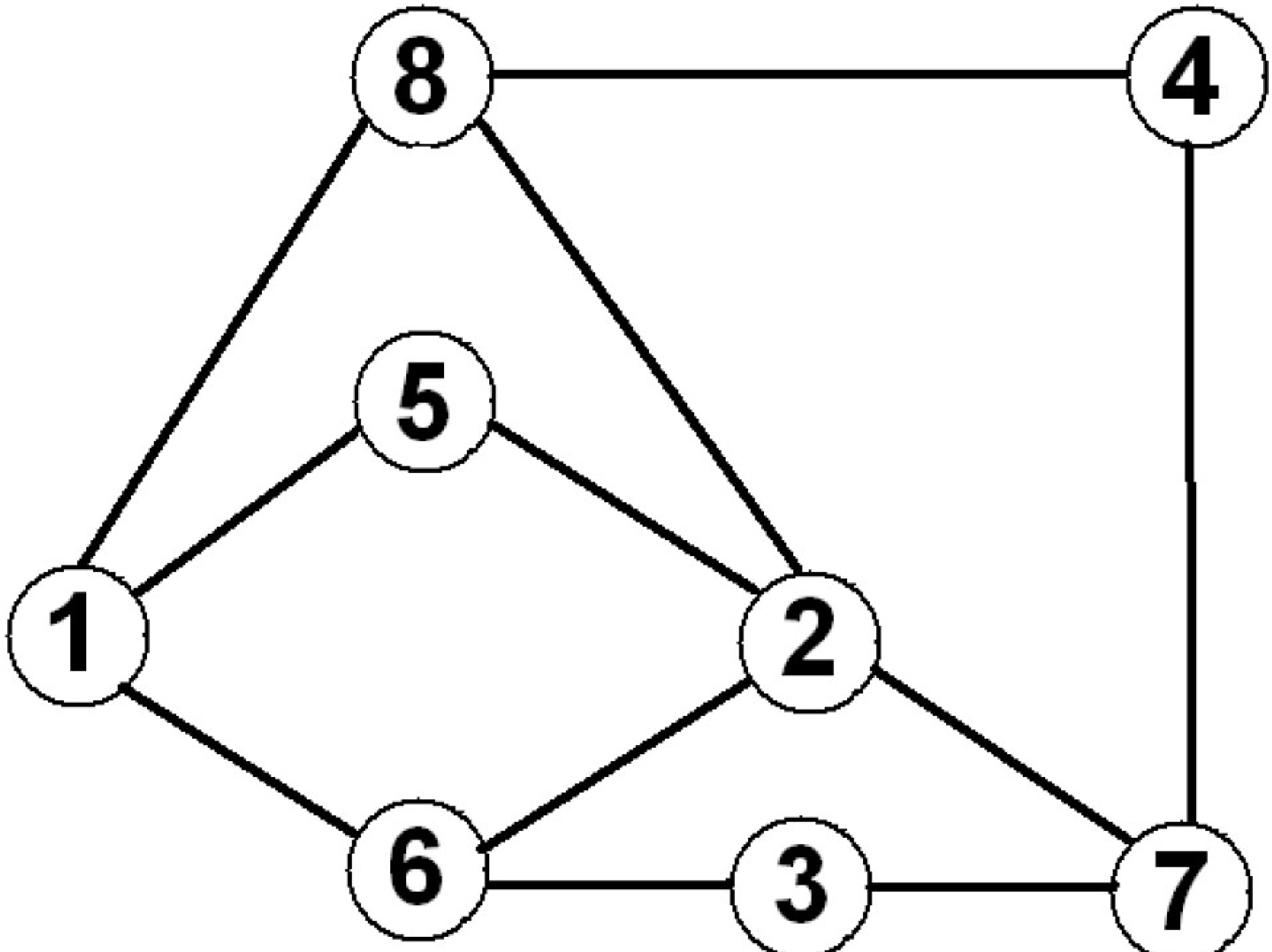
NP-Completeness

- How do I prove that problem X is NP-complete?
  - Show that candidate solutions for X can be checked in polynomial time;
- Show that there exists an NP-complete problem Y such that an algorithm that solves X can also solve Y. This is called a reduction.
- A reduction establishes that a fast solution for X would also give a fast solution for Y.
- The first established NP-complete problem was boolean satisfiability, or SAT. This is called Cook's Theorem (1971).
- If your problem is NP-complete, you can safely give up looking for an efficient, exact algorithm.

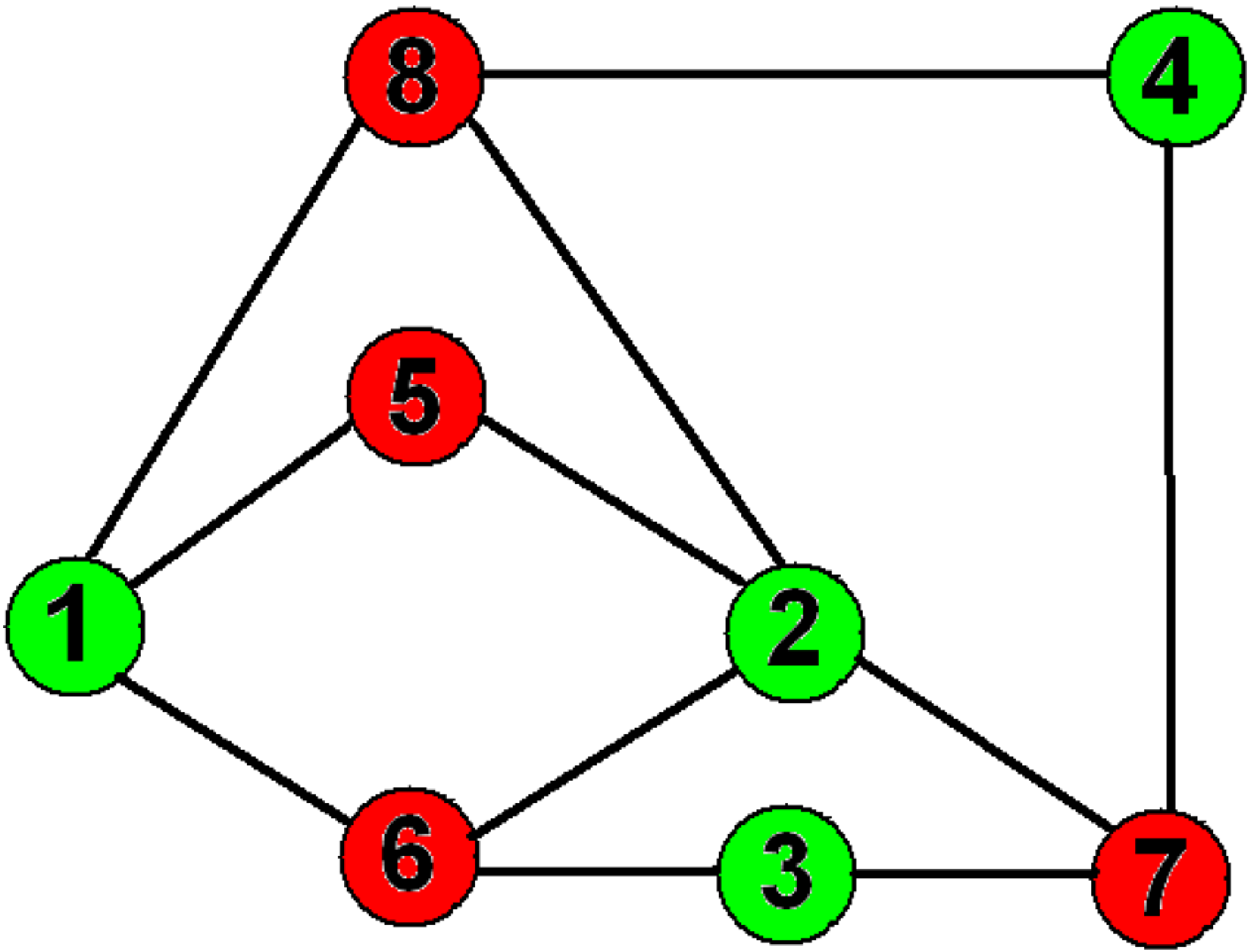
Graph Coloring

- Given an undirected graph, color each vertex such that no two vertices of the same color share an edge. What is the fewest number of colors that can be used?
- Checking “Is a graph colorable using 2 colors?” is easy, and solvable in polynomial time.
- Checking “Is a graph colorable using 3 colors?” is NP-complete.

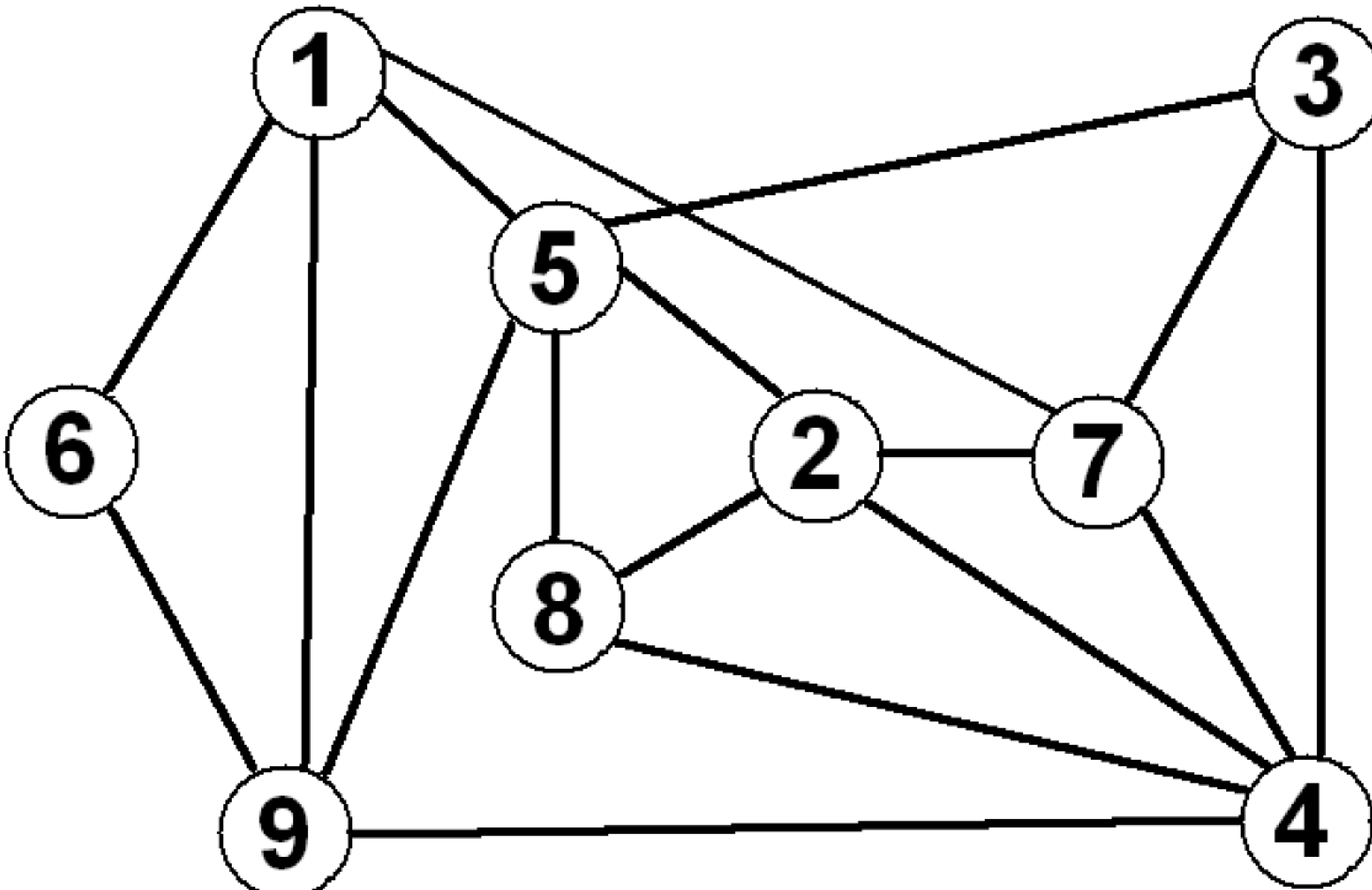
Example: 2-colorable Graph



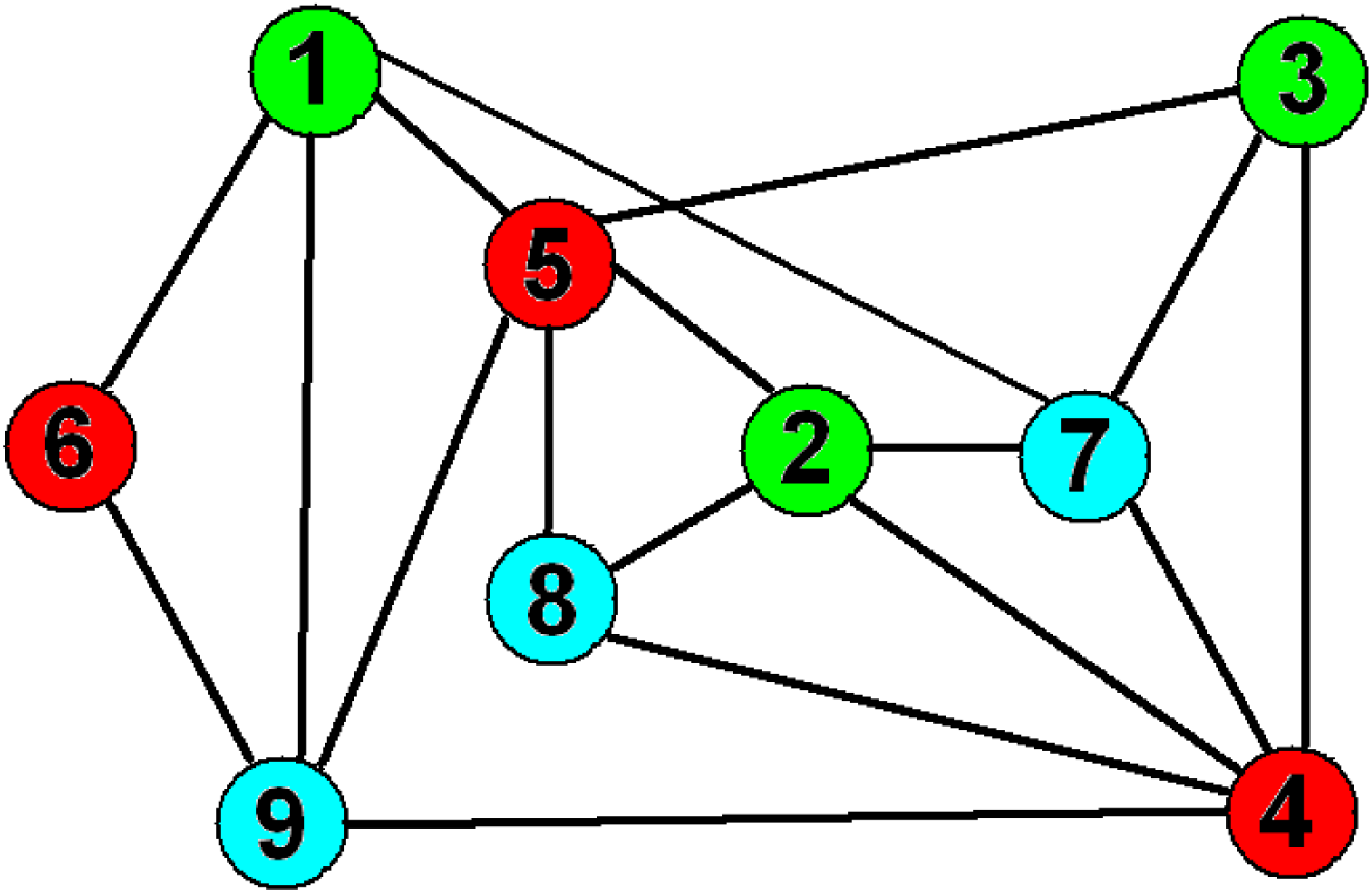
Example: 2-colorable Graph



Example: 3-colorable Graph



Example: 3-colorable Graph



Undecidability

- Even worse than NP-complete, undecidable problems are those for which no algorithm can exist that is guaranteed to always solve it correctly.
- Example 1: the halting problem: “Will my problem ever stop executing?”
- Example 2: Kolmogorov complexity: “What is the simplest program that can generate a given string?”
- Problems can be shown to be undecidable by using similar reductions as for NP-completeness proofs.

Graph Traversal – BFS and DFS

http://eecourses.technion.ac.il/044268/

To learn more...

Books:

- Harel “Algorithmics” (2004)
  - accessible and easy to read
- Cormen et al. “Introduction to Algorithms” (2001)
  - The comprehensive algorithms “bible”
  - Often abbreviated as CLR or CLRS
- Garey and Johnson “Computers and Intractability: A Guide to the Theory of NP-Completeness (1979)

To learn more...

Courses: CPSC 320, 421, 500, 506

People: BETA Lab

These slides:

`http://www.cs.ubc.ca/`
`    ~ankgupta/refresher2011.html`