CPSC 303 Feb 7

Today: DN diff $\underbrace{\quad}_{\text{bump into}}$ interpolation...

Recall: Lagrange Interpolation:

Have data: (x_0, y_0) (x_1, y_1) (x_2, y_2)

$(2, y_0)$ $(3, y_1)$ $(7, y_2)$

We want poly $p(x) = c_0 + c_1 x + c_2 x^2$ s.t.

$p(2) = y_0$, $p(3) = y_1$, $p(7) = y_2$

Answer:

$$p(x) = y_0 \underbrace{L_0(x)}_{\frac{(x-3)(x-7)}{(2-3)(2-7)}} + y_1 \underbrace{L_1(x)}_{\frac{(x-2)(x-7)}{(3-2)(3-7)}} + y_2 \underbrace{L_2(x)}_{\frac{(x-2)(x-3)}{(7-2)(7-3)}}$$

x^2 term $\overset{?}{=}$ 1

$$\boxed{\begin{array}{l} L_0(3) = 0,\ L_0(7) = 0 \\ L_0(2) = 1 \end{array}}$$

unique poly deg 2 s.t.

polys of deg 2

$$p(2) = y_0 \underbrace{L_0(2)}_{1} + y_1 \underbrace{L_1(2)}_{0} + y_2 \underbrace{L_2(2)}_{0}$$

$c_0 + c_1 X + c_2 X^2$ $\qquad$ L_0, L_1, L_2

Linear Combos $(1, x, x^2)$ $\qquad$ linear combos $(L_0(x), L_1(x), L_2(x))$

X^2 term coeff: $\boxed{y_0 \frac{1}{(2-3)(2-7)} + y_1 \frac{1}{(3-2)(3-7)} + y_2 \frac{1}{(7-2)(7-3)}}$

Fact: To fit $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ with $c_0 + c_1 X + c_2 X^2$

$c_2 =$ $\qquad$ for $x_0 = 2,\ x_1 = 3,\ x_2 = 7$

In general

$$\boxed{c_2 = \frac{y_0}{(x_0-x_1)(x_0-x_2)} + \frac{y_1}{(x_1-x_0)(x_1-x_2)} + \frac{y_2}{(x_2-x_0)(x_2-x_1)}}$$

$$y_0 L_0(x) = y_0 \underbrace{\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}}_{X^2 \text{ term}}$$

$\leftarrow X^2$ term is $\underbrace{(x-x_1)(x-x_2)}_{\text{is } X^2}$

symmetric in permuting 0,1,2

$$\boxed{c_2 = \frac{y_0}{(x_0-x_1)(x_0-x_2)} + \boxed{\frac{y_1}{(x_1-x_0)(x_1-x_2)}} + \frac{y_2}{(x_2-x_0)(x_2-x_1)}}$$

$0 \to 1$
$1 \to 0$
$2 \to 2$

$$\boxed{\frac{y_1}{(x_1-x_0)(x_1-x_2)}} + \cdots$$

Divided differences: Function, f, then $x_0, x_1, \dots, x_n \in \mathbb{R}$

$$f[x_0] \overset{\text{def}}{=} f(x_0)$$ "0th divided diff"

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

"1st div diff"

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$ "2nd divided difference"

Facts: Properties of divided diffs:

(1) Symmetry: $f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

$$f[x_1, x_0] = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

(2) $f[x_0, x_1, x_2] = f[x_1, x_0, x_2] = f[x_0, x_2, x_1] = \dots$

$$f[x_0, x_1, x_2] := \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{\left(\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}\right)}{x_2 - x_0} =$$ 

$$= \frac{f(x_2)}{(x_2 - x_1)(x_2 - x_0)} + f(x_1) \qquad + \frac{f(x_0)}{(x_1 - x_0)(x_2 - x_0)}$$

$f(x_2)$ term

$$\longleftarrow \frac{\left\{ \frac{-1}{x_2-x_1} - \frac{1}{x_1-x_0} \right.}{x_2-x_0} = \frac{-\frac{(x_1-x_0)+(x_2-x_1)}{(x_2-x_1)(x_1-x_0)}}{x_2-x_0}$$

$$\parallel$$

$$\frac{-1}{(x_2-x_1)(x_1-x_0)} \;\overset{\longleftarrow}{=}\; \frac{-\left(\frac{\cancel{x_2-x_0}}{(x_2-x_1)(x_1-x_0)}\right)}{\cancel{x_2-x_0}}$$

cancels

$$\parallel$$

$$\frac{1}{(x_1-x_2)(x_1-x_0)}$$

Not only:

$$f[x_0,x_1,x_2] = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)} + \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}$$

$=$ c_2 term of $c_0+c_1x+c_2x^2$ in Lagrange expression, the unique deg ≤ 2 poly $p(x)$

$p(x) = c_0+c_1 x + c_2 x^2$ s.t.
$$p(x_0) = f(x_0)$$
$$p(x_1) = f(x_1)$$
$$p(x_2) = f(x_2)$$

So, i.e.

$$p(x) = c_0 + c_1 x + f[x_0,x_1,x_2]\, x^2$$