CPSC 303, Feb 24

	Su	M	W	F	Sa
Feb		24	26	28	29
Mar	1	2	4	6	
		9	11		

HW 5 (solutions Friday morning)

HW 6

Sample Midterm Questions

Give solutions immediately

Midterm

Midterm covers up to end of Ch 10, to be done by Feb 26

1 2-sided $8\frac{1}{2}'' \times 11''$ sheet of notes

Finish Ch 10:

§ 10.6 Chebyshev interpolation

§ 10.7 Interpolation when $x_0, \ldots, x_n$ are not distinct;

esp. Hermite cubic interpolation

$$x_0 = x_1 = t_0 \quad , \quad x_2 = x_3 = t_1$$

(also be used in Ch 11)

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§ 10.6 "Game": You get to choose any

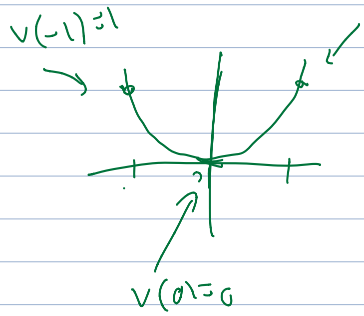
$x_0, x_1, \ldots, x_n \in \mathbb{R}$, but you want

$$\mathrm{Max}(x_0, \ldots, x_n) = \max_{x \in [-1,1]} |x - x_0|\,|x - x_1| \cdots |x - x_n|$$

to be as small as possible ...

For $n=2$: x_0, x_1 best = probably $\pm 1/\sqrt{2}$

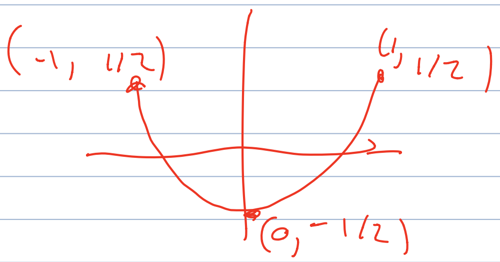
(is true)



$x_0 = x_1 = 0$

$$(x-x_0)(x-x_1) = x^2 = v(x)$$

better $v(x) = x^2 - 1/2$



turns out to be the best choice of x_0, x_1

similarly: there is a $v(x) = x^n +$ lower order

s.t. on $x \in [-1,1]$, $|v(x)| \le \left(1/2\right)^{n-1}$

"Reason"

$$\cos(0 \cdot \vartheta) = 1$$

$$\cos(\vartheta) = \cos\vartheta$$

$$\cos(2\vartheta) = 2\cos^2\vartheta - 1$$

claim

$$\cos(n\vartheta) = T_n(\cos\vartheta), \text{ where}$$

T_n is poly of degree n with leading coef 2^{n-1}!

(e.g. wikipedia page, ... "Chebyshev polynomials of the first kind"

$$\cos 3\vartheta = 4\cos^3\vartheta - 3\cos\vartheta$$

$$\cos 4\vartheta = 8\cos^4\vartheta - 8\cos^2\vartheta + 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_2(x) = 2x^2 - 1 = 2(x^2 - 1/2) = 2\left(x + \tfrac{1}{\sqrt{2}}\right)\left(x - \tfrac{1}{\sqrt{2}}\right)$$

Claim: For any $x \in [-1, 1]$, $x = \cos\vartheta$ for some ϑ

$\Rightarrow$ $\cos(n\vartheta) = T_n(x)$ is at most 1 in absolute value

So ... it turns out that

$$\frac{T_n(x)}{2^{n-1}} = x^n + \text{lower terms}$$

and $-1 \le x \le 1$ $\Longrightarrow$ $\left|\frac{T_n(x)}{2^{n-1}}\right| \le \frac{1}{2^{n-1}}$.

$$T_{n+2}(x) = \bigcirc\, T_{n+1}(x) + \bigcirc\, T_n(x)$$ (3 term recurrence)

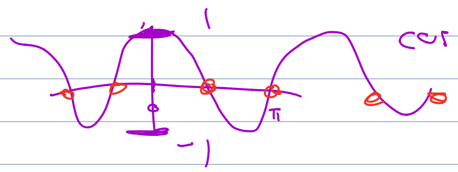
coefs

Standard: $(x-x_0)\cdots(x-x_{n-1})$ deg n

$= x^n +$ lower stuff

e.g. $(x^2 - 1/2) = (x - 1/\sqrt{2})(x + 1/\sqrt{2})$

$= \frac{1}{2}(2x^2 - 1)$



$\cos \vartheta = x$

$\vartheta = (\arccos)(x)$

$T_n(x) = \cos(n\vartheta)$

for any $x \in [-1, 1]$
there is some ϑ

$|T_n(x)| \leq 1$

When is $2x^2 - 1 = 0$?

(1) $x = \pm 1/\sqrt{2}$

(2) $2x^2 - 1 = T_2(x)$, $T_2(\cos\vartheta) = \cos(2\vartheta)$

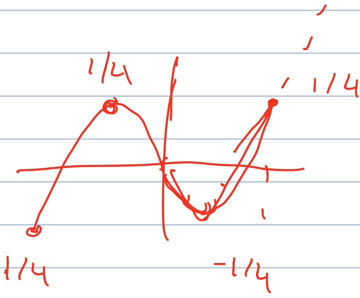
whenever
$\cos(2\vartheta) = 0$

$T_2(\cos\vartheta) \Leftarrow 0$

$\frac{1}{2} T_2(x)$

$\pm 1/\sqrt{2}$

$\frac{1}{4} T_3(x)$



$T_3(x) = 4x^3 - 3x$

Recall: for any $x_0, \ldots, x_n$ distinct:

Say f is as differentiable as we want,

$f, f', f'', \ldots$ are "reasonably bounded"

e.g. $f(x) = \sin(x)$, $f(x) = \cos(x)$, $f(x) = e^x$

$f' = \cos(x)$ $\quad\vdots\quad$ e^x

$f'' = -\sin(x)$ $\quad\vdots$

$\vdots$

We know

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0) + \ldots$$

$$f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

we know

$$p(x) = f(x) \quad \text{for} \quad x = x_0, x_1, \ldots, x_n$$

error

$$f(x) - p(x) = (x - x_0)(x - x_1)\ldots(x - x_n)\frac{f^{(n+1)}(\xi)}{(n+1)!}$$

for some ξ on any interval containing $x, x_0, \ldots, x_n$

So

$$|f(x) - p(x)| = V(x)\frac{\text{bound on } (n+1)^{st} \text{ der of } f}{(n+1)!}$$

$$V(x) = (x - x_0)\ldots(x - x_n)$$

Taking $x_0, \ldots, x_n$ to be the roots of $T_{n+1}(x)$ gives smallest $\max_{x\in[-1,1]} |(x-x_0)\cdots(x-x_n)|$

and is called Chebyshev Interpolation = § 10.6 title

We also allow

$$(x-x_0)(x-x_1) = (x+1.1)(x-1.3)$$

but $\max_{x\in[-1,1]} |(x+1.1)(x-1.3)| > \frac{1}{2}$