Marks

[10] **1.** Give a formal description of a Turing machine—and explain how your machine works—that takes as input, $x \in \{0,1\}^*$, and (1) accepts x if it has an even number of 1's, and (2) rejects x if not. You should **explicitly write** your choice of $Q, \Sigma, \Gamma, \delta, q_0, q_{\mathrm{accept}}, q_{\mathrm{reject}}$. For example, you should write $\Sigma = \{0,1\}$, since this is the input alphabet.

Answer: There are many possibilities. Here is one (which is what most people did). Move from left to right through the input, and have one state express the fact that we have seen an odd number of ones, say q_1, and another for an even number of states. One can have the even state just be q_0.

Formally: $Q = \{q_0, q_1, q_{\mathrm{accept}}, q_{\mathrm{reject}}\}$**,** $\Sigma = \{0,1\}$**,** $\Gamma = \{0, 1, \mathit{blank}\}$**, and** δ**'s values given by**

$$\delta(q_0, 0) = (q_0, 0, R), \quad \delta(q_0, 1) = (q_1, 0, R), \quad \delta(q_0, \mathit{blank}) = (q_{\mathrm{accept}}, 0, R),$$

$$\delta(q_1, 0) = (q_1, 0, R), \quad \delta(q_1, 1) = (q_0, 0, R), \quad \delta(q_1, \mathit{blank}) = (q_{\mathrm{reject}}, 0, R).$$

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[16] **2.** (4 points for each part) Briefly justify your answers:

(a) Let $\mathcal{P}$ and $\mathcal{I}$ be any countably infinite sets and let "Result" be a function described in Axiom 1. Are there languages, i.e., subsets of $\mathcal{I}$, that are not recognized by any element of $\mathcal{P}$?

Answer: Since $\mathcal{I}$ is countably infinite, the set of languages is uncountable, and hence any map from the programs to the set of languages cannot be surjective. Hence there are languages not recognized by any program.

(b) Russell's paradox involves considering "the set of all sets that do not contain themselves." What is the standard way of resolving this paradox?

Answer: The collection of sets with a some properties–including the collection of all sets–is defined to be a "class" (which may be a set or something "larger").

Remark: One often thinks intuitively that a set shouldn't contain itself, but this is not strictly necessary. However, a collection of sets as large as those "that do not contain themselves" should not be a set (as can be seen by Russell's paradox). There are collections of sets that are sets, such as the power set of a set. However, we still want to be able to speak about collections of sets such as all sets, so some of these collections are "classes" but not "sets."

(c) Assume that you have a set of axioms in a system of logic where you can construct a sentence, S, whose meaning is "S is not provable (from the axioms)." Assume that you have an interpretation of your sentences so that each sentence is either true or false, but not both. Show that at least one of the following holds: (1) Some sentence that is provable (from the axioms) is false, or (2) Some sentence is true but not provable (from the axioms).

Answer: S is either true or false. If S is false, then—given the meaning of S—S is provable (case (1)). Is S is true, then, similarly, S is not provable but true (case (2)).

Remark: (1) can definitely hold: consider the case where the axioms consist of all sentences (whether they are true or false under our interpretation).

(d) In class we showed that $|S| < |Power(S)|$ for any set S, where $Power(S)$ is the set of all subsets of S. Our proof involved considering any map $f\colon S \to Power(S)$, and forming the set

$$T = \{s \in S \mid s \notin f(s)\}.$$

How do we use this T and what assumptions do we make on f to show that $|S| < |Power(S)|$?

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Answer: If not, then there is a surjection $f\colon S \to Power(S)$. Then, with T as above, since f is a surjection there is a $t \in S$ such that $f(t) = T$. But then either (i) $t \in T$, or (ii) $t \notin T$; if $t \in T$, then by definition of T we have $t \notin f(t) = T$, which is a contradition; similarly, if $t \notin T$, then by definition of T we have $t \in f(t) = T$, again a contradiction. Hence the assumption that such an f—i.e., a surjection—exists leads to a contradiction, and hence $|S| < |Power(S)|$.

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[10] **3.** Let NO-COUNT-3 be the language of descriptions, $\langle p, i \rangle$, consisting of a 421Simple program, p, and an input, i, such that p has a variable named COUNT, and when p is run on input i, the variable COUNT that never attains the value 3 (at any point in its computation). Is NO-COUNT-3 decidable (say, by a 421Simple program) or recognizable? Justify your answer.

Answer: NO-COUNT-3 is not recognizable (hence not decidable). Indeed, the complement of NO-COUNT-3 can be recognized by a universal machine that simulates p on input i (unless the input is not of the form $\langle p, i \rangle$, in which case we accept) and accepts if COUNT ever attains the value 3. NO-COUNT-3 cannot be decided, or else we could solve the Halting Problem by taking an input of the form, $\langle p, i \rangle$, renaming the counter COUNT of p if it exists, and replace every END statement in p by a set of statements that sets COUNT to 3 (one RESET and three AUG statements). Hence the complement of NO-COUNT-3 is recognizable but not decidable, and hence NO-COUNT-3 is not recognizable.

A lot of people said that a universal machine could recognize NO-COUNT-3, rather than its complement. The point is that if the simulation of the universal machine (that is looking for COUNT to be set to 3) never halts, then the simulated program never takes on the value 3. Hence the univesal machine accepts the complement of NO-COUNT-3.

Remark: When we speak of the complement of NO-COUNT-3, we mean (i) either strings that not of the form $\langle p, i \rangle$, or (ii) those that are of the form $\langle p, i \rangle$ that are not elements of NO-COUNT-3. The case (i) is easy to decide, so we usually just speak of case (ii).

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Be sure that this examination has 6 pages including this cover

The University of British Columbia

Midterm Examinations - October 2014

Computer Science 421/501

Closed book examination | Time: 50 minutes

Name ______________________ **Signature** ______________________

Student Number ______________________ **Instructor's Name** ______________________

Section Number ______________________

Special Instructions:

Calculators, notes, or other aids may not be used. Answer questions on the exam. This exam is two-sided!

Rules governing examinations

1. Each candidate should be prepared to produce his library/AMS card upon request.

2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

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2		16
3		10
Total		36