

Homework 2, CPSC 421/501, 2024

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(2) 9.2.6

(a) Yes — let's prove the contrapositive:

if f not injective, then for some

$a_1, a_2 \in A$, $a_1 \neq a_2$ and $f(a_1) = f(a_2)$.

Hence $g(f(a_1)) = g(f(a_2))$. Hence

$$\begin{aligned}(gf)(a_1) &= g(f(a_1)) = g(f(a_2)) \\ &= (gf)(a_2)\end{aligned}$$

So gf is not injective.

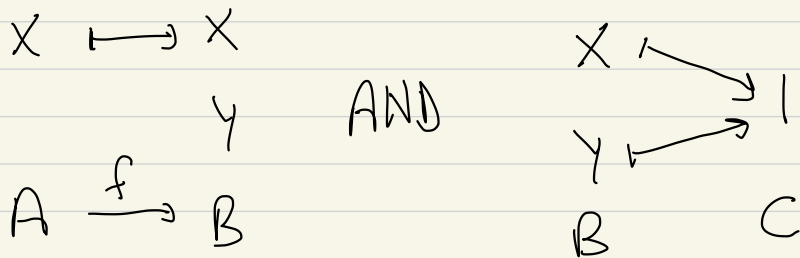
Hence, if gf is injective,

then f is injective.

(b) No: If $A = \{x\}$,

$B = \{x, y\}$, and $C = \{1\}$,

and f and g are given by



then gf takes x to 1 ,

and hence gf is injective.

But $g(x) = g(y) = 1$, so g

is not injective.

(c) Yes: by (a), f is injective.

Hence for each $b \in \text{Image}(f)$ there

is a unique $a \in A$ st. $f(a) = b$.

Hence $|\text{Image}(f)| = |A| \geq |B|$

$\text{Image}(f) \subset B$ and $|\text{Image}(f)| \geq |B|$,

we have $\text{Image}(f) = B$.

If g were not injective, then

for some $b_1 \neq b_2$ with $b_1, b_2 \in B$

we would have $g(b_1) = g(b_2)$.

Since $\text{Image}(f) = B$, we have

$$f(a_1) = b_1 \quad \text{and} \quad f(a_2) = b_2$$

for some $a_1, a_2 \in A$, and

$a_1 \neq a_2$ since $f(a_1) \neq f(a_2)$.

But then

$$(gf)(a_1) = g(b_1) = g(b_2) = (gf)(a_2)$$

but $a_1 \neq a_2$, so gf is not surjective.

(3) 9.2.8

(a) $f(A) = \emptyset$, and

$$f(C) = \{A, B, C\} = P$$

(b) Since $A \notin f(A)$, $A \in T$

Since $C \in f(C)$, $C \notin T$

We know $T \neq f(A)$, since $A \notin f(A)$

but $A \in T$. We know $T \neq f(C)$ since

$C \in f(C)$ but $C \notin T$.

(c) Now we know $f(B) = \{A, C\}$, and

have $B \notin f(B)$ so $B \in T$.

Since $C \notin T$ and $A, B \in T$, we have

$$T = \{A, B\}.$$

(4) 9.2.13

(a) Each of Johnny, Moira, Alexis loves themselves. David does not love themselves. Hence $T = \{\text{David}\}$.

(b) If David does not love themselves, then $\text{David} \in T$ and $\{\text{People Whom David Loves}\} = \emptyset$.

So $\text{David} \notin \{\text{People Whom David Loves}\}$

but $\text{David} \in T$

so $T \neq \{\text{People Whom David Loves}\}$.

(5) 9.4.1 (a)

if $(i, j) \in \mathbb{N}^2$ and $i + j \leq 2$,

then $(i, j) = (1, 1)$;

if $(i, j) \in \mathbb{N}^2$ and $i + j = 3$,

then $(i, j) = (1, 2)$ or $(2, 1)$

Similarly,

if $(i, j) \in \mathbb{N}^2$ and $i + j = k$ for

some $k = 2, 3, 4, \dots$ then

(i, j) is one of finitely many

values, namely

$(1, k-1)$ or $(2, k-2)$ or \dots or $(k-1, 1)$

Hence, for each $k = 2, 3, 4, \dots$
there are only finitely many
 $(i, j) \in \mathbb{N}^2$ such that $i+j = k$.

(And, of course each $(i, j) \in \mathbb{N}^2$
does satisfy $i+j = k$ for some $k = 2, 3, \dots$)

Hence we can write a list of all elements
of \mathbb{N}^2 , namely

$\underbrace{(1, 1)}_{i+j=2}, \quad \underbrace{(1, 2), (2, 1)}_{i+j=3}, \quad \underbrace{(1, 3), (2, 2), (3, 1)}_{i+j=4},$

\dots

Hence \mathbb{N}^2 is in bijection with \mathbb{N} .

[Note: There are many equivalent ways of writing this; for example, one can write $\mathbb{N} \times \mathbb{N}$ as a table

(1,1) (1,2) (1,3) ...

(2,1) (2,2) (2,3) ...

(3,1) (3,2) (3,3) ...

⋮ ⋮ ⋮ ⋮

Then the solution above is just like the proof that \mathbb{Q}^+ is countable.]