Homework 2, CPSC 421/501, 2024

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(2) 9.2.6

(a) Yes — let's prove the contrapositive:

if f not injective, then for some

$a_1, a_2 \in A$, $a_1 \neq a_2$ and $f(a_1) = f(a_2)$.

Hence $g(f(a_1)) = g(f(a_2))$. Hence

$$(gf)(a_1) = g(f(a_1)) = g(f(a_2))$$

$$= (gf)(a_2)$$

So gf is not injective.

Hence, if gf is injective,

then f is injective.

(b) No: If $A = \{x\}$, $B = \{x, y\}$, and $C = \{1\}$, and f and g are given by

$$\begin{array}{ccc} x & \longmapsto & x \\ & & y \\ A & \xrightarrow{f} & B \end{array} \qquad \text{AND} \qquad \begin{array}{ccc} x & \searrow & \\ & & 1 \\ y & \nearrow & \\ B & & C \end{array}$$

then gf takes x to 1, and hence gf is injective. But $g(x) = g(y) = 1$, so g is not injective.

(c) Yes! by (a), f is injective.

Hence for each $b \in \text{Imag}(f)$ there is a unique $a \in A$ s.t. $f(a) = b$.

Hence $|\text{Image}(f)| = |A| \geq |B|$

$\text{Image}(f) \subset B$ and $|\text{Image}(f)| \geq |B|$, we have $\text{Image}(f) = B$.

If g were not injective, then for some $b_1 \neq b_2$ with $b_1, b_2 \in B$ we would have $g(b_1) = g(b_2)$.

Since $\text{Image}(f) = B$, we have

$f(a_1) = b_1$ and $f(a_2) = b_2$

for some $a_1, a_2 \in A$, and

$a_1 \neq a_2$ since $f(a_1) \neq f(a_2)$.

But then

$$(gf)(a_1) = g(b_1) = g(b_2) = (gf)(a_2)$$

but $a_1 \neq a_2$, so gf is not surjective.

(3) 9.2.8

(a) $f(A) = \emptyset$, and

$f(C) = \{A, B, C\} = P$

(b) Since $A \notin f(A)$, $A \in T$

Since $C \in f(C)$, $C \notin T$

We know $T \neq f(A)$, since $A \notin f(A)$

but $A \in T$. We know $T \neq f(C)$ since

$C \in f(c)$ but $C \notin T$.

(c) Now we know $f(B) = \{A, C\}$, and

have $B \notin f(B)$ so $B \in T$.

Since $C \notin T$ and $A, B \in T$, we have

$T = \{A, B\}$.

(4) 9.2.13

(a) Each of Johnny, Moira, Alexis loves themself. David does not love themself. Hence $T = \{\text{David}\}$.

(b) If David does not love themself, then David $\in T$ and

$\{\text{People Whom David Loves}\} = \emptyset$.

So David $\notin \{\text{People Whom David Loves}\}$

but David $\in T$

so $T \neq \{\text{People Whom David Loves}\}$.

(5) 9.4.1 (a)

if $(i,j) \in \mathbb{N}^2$ and $i+j \leq 2$,

then $(i,j) = (1,1)$;

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if $(i,j) \in \mathbb{N}^2$ and $i+j = 3$,

then $(i,j) = (1,2)$ or $(2,1)$

Similarly,

if $(i,j) \in \mathbb{N}^2$ and $i+j = k$ for

some $k = 2,3,4,\ldots$ then

(i,j) is one of finitely many

values, namely

$(1,k-1)$ OR $(2,k-2)$ OR ... OR $(k-1,1)$

Hence, for each $k = 2, 3, 4, \ldots$

there are only finitely many

$(i, j) \in \mathbb{N}^2$ such that $i + j = k$.

(And, of course each $(i, j) \in \mathbb{N}^2$
does satisfy $i + j = k$ for some $k = 2, 3, \ldots$)

Hence we can write a list of all elements of $\mathbb{N}^2$, namely

$$\underbrace{(1,1)}_{i+j=2}, \quad \underbrace{(1,2), (2,1)}_{i+j=3}, \quad \underbrace{(1,3), (2,2), (3,1)}_{i+j=4},$$

...

Hence $\mathbb{N}^2$ is in bijection with $\mathbb{N}$.

[Note: There are many equivalent ways of writing this; for example, one can write $\mathbb{N} \times \mathbb{N}$ as a table

$$\begin{array}{cccc} (1,1) & (1,2) & (1,3) & \cdots \\ (2,1) & (2,2) & (2,3) & \cdots \\ (3,1) & (3,2) & (3,3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Then the solution above is just like the proof that $\mathbb{Q}^+$ is countable.]