CPSC 421 Solutions to HW 3, 2024

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(2) There are many ways to do this...
Here's one:

to each $(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k$, say the "rank" of $(n_1, \ldots, n_k)$ is $n_1 + n_2 + \ldots + n_k$. Then

(1) each element of $\mathbb{N}^*$ has finite "rank", and

(2) for each $r = 0, 1, 2, \ldots$ there are only finitely many elements of $\mathbb{N}^*$ of "rank" equal to r: indeed,

rank $(n_1, \ldots, n_k) = r$ implies

(a) $k \leq r$ (since each element of $\mathbb{N} = \{1, 2, \ldots\}$ has value ≥ 1), and

(b) each n_i has $1 \leq n_i \leq r$.

In view of (a), (b), we have

rank $(n_1, \ldots, n_k) \leq r$ implies

$(n_1, \ldots, n_k)$ lies in

$S_r = [r]^0 \cup [r]^1 \cup \ldots \cup [r]^r$, where

$[r] = \{1, 2, \ldots, r\}$, so S_r is finite.

Hence we can list the elements of $\mathbb{N}^*$ by their rank, listing them

as the elements of S_0

" " " S_1

" " " S_2

⋮

giving a list $l_1, l_2, l_3, l_4, \ldots$

s.t. each element of $\mathbb{N}^*$ is found somewhere on the list. Hence this gives a surjection

$$\mathbb{N} \longrightarrow \mathbb{N}^* \quad (\text{namely } i \mapsto l_i).$$

Hence $\mathbb{N}^*$ is countable.

(3) For example:

$a \in LangRecBy_{Duck}(duck1)$

Hence

$duck1 \circ_0 a = \{ p \circ_0 i \mid p$ is a valid Duck program s.t. p accepts $i \}$

$= ACCEPTANCE_{Duck}$.

By contrast:

$a \notin LangRecBy_{Duck}(duck2)$

Hence

$duck2 \circ_0 a \notin ACCEPTANCE$

(b) The set of languages recognizable by Duck programs is the set of languages of the form

$$\Sigma^{k_1} \cup \Sigma^{k_2} \cup \dots \cup \Sigma^{k_m}$$

s.t. $m \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \dots\}$

and each $k_i \in \mathbb{Z}_{\geq 0}$.

However duck1 σ_0 a and duck2 σ_0 a are both of length 7, one lying in $\text{ACCEPTANCE}_{\text{Duck}}$, the other no. Hence $\text{ACCEPTANCE}_{\text{Duck}}$ is not Duck-recognizable.