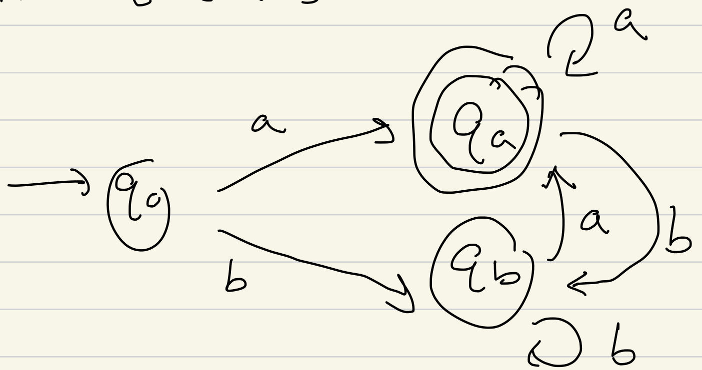
CPSC 421/501 HW Sol 6, 2024

1. Joel Friedman

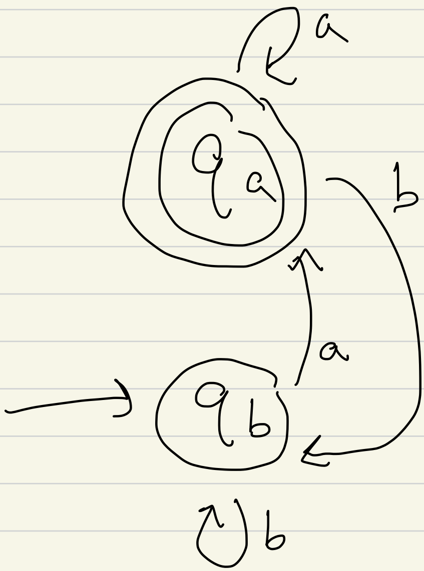
2 (a) Perhaps the most natural DFA has 3 states:



which rejects the empty string ε in q_0, and then transitions to two states, a state q_a for when the last symbol read is an a, the other for b, q_b

However, we don't really need q_0,

since we can equivalenty start in q_b (which then rejects the empty string). Hence the two-state DFA



also recognizes L.

(b) We similarly build a DFA that remembers the last 2 letters, with states

$Q = \{q_{aa}, q_{ab}, q_{ba}, q_{bb}\}$

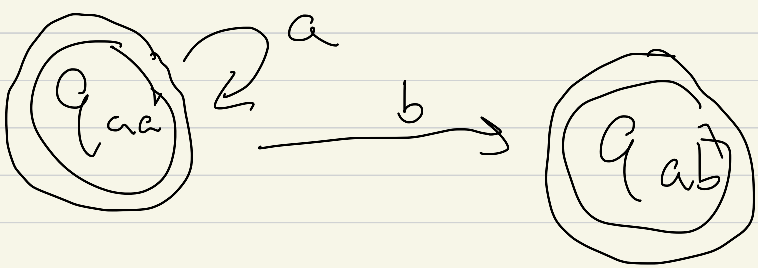
where q_{xy} means that the last two symbols are xy.

If the 2nd to last symbol is "a" then we accept, so the accepting (or "final") states are

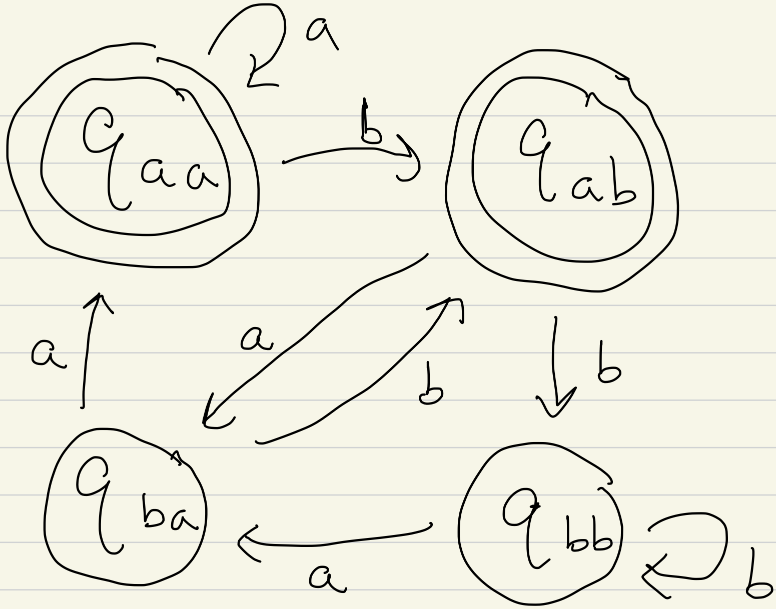
$F = \{q_{aa}, q_{ab}\}$.

If we are is state q_{aa} and

we read an "a", then the new last two symbols are still aa, whereas if we read a "b" then the new two last symbols are ab. Hence we have the transitions

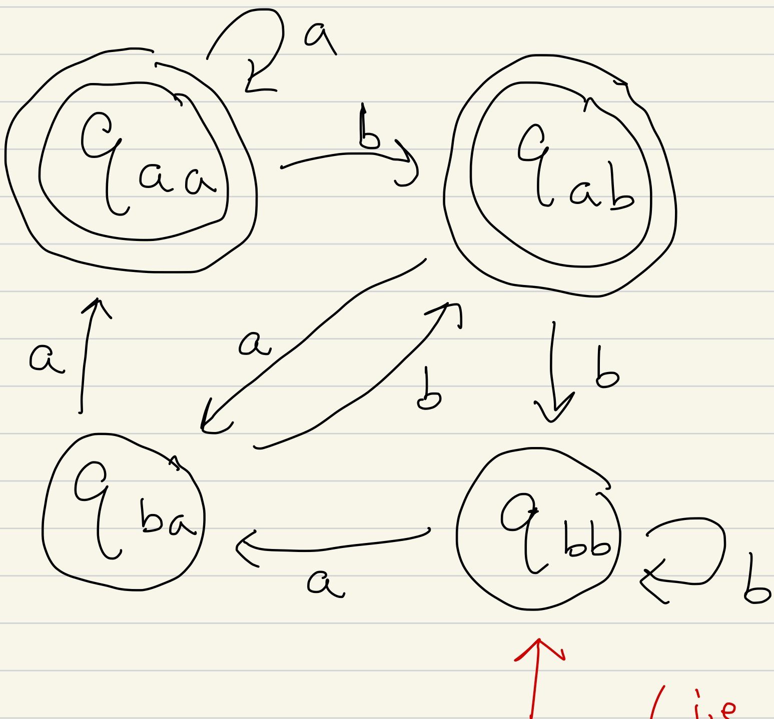


doing similarly for the other 3 states we get a partial DFA



Now we see that we can take q_{bb} as the initial state, because doing so will reject ε, a, b and after two steps through the DFA we get to the state that appropiately represents the last two symbols we've seen.

Hence a 4-state DFA accepting L is



(i.e. q_{bb} is the initial state)

(3) 6.1.2

(a) Say that

$$\forall n \geq n_0', \qquad a^n \in L \iff a^{n+m'} \in L$$

and

$$\forall n \geq n_0, \qquad a^n \in L \iff a^{n+m} \in L$$

then

$\forall n \geq \max(n_0', n_0)$

$$a^{n+m} \in L \iff a^n \in L \iff a^{n+m'} \in L$$

so

$$a^{n+m} \in L \iff a^{n+m'} \in L$$

$$\iff a^{n+m+(m'-m)} \in L$$

Setting $k = n+m$, we have

$$a^k \in L \iff a^{k+(m'-m)} \in L$$

for all k s.t. $n = k - m \geq \max(n_0', n)$,

i.e., for $k \geq C$, where $C = m + \max(n_0', n)$.

Hence L is eventually $(m' - m)$-periodic.

(b) Say that L is p'-periodic. We may write $p' = p \cdot r + (p' \bmod p)$ (where $r = \lfloor p'/p \rfloor$)

i.e. $p' = p \cdot r + i$ where $0 \leq i \leq p-1$.

Since L is p'-periodic and p periodic, we have

L is $\begin{cases} (1)\ p' - p \text{ periodic} & (\text{if } p' - p \geq 1), \text{ hence} \\ (2)\ p' - 2p \quad " & (\ " \ p' - 2p \geq 1), \text{ hence} \end{cases}$

(3) $p' - 3p$ " (" $p' - 3p \geq 1$), hence

⋮

and (by induction on r)

⋮

$(p' - rp)$ periodic (" $p' - rp \geq 1$).

So if $p' \bmod p = i$ is one of $1, 2, \ldots, p-1$

then L is i-periodic, which is impossible

since $1 \le i < p$ and p is the periodicity

of L.

(c) If M is a DFA that recognizes L,

and the cycle length of M is p', then

L must be p'-periodic. Hence (b) implies

that p' is divisible by p.

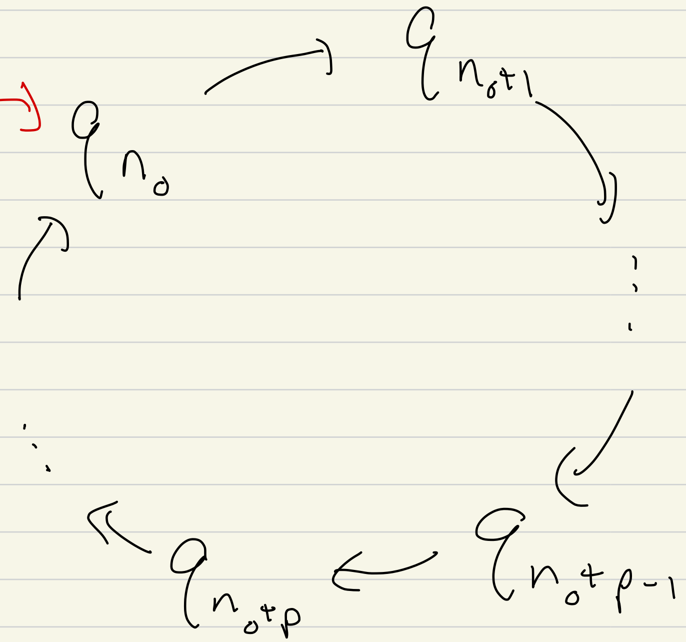
(d) Say that the cycle in the

DFA has length p' and

$p' > p$. Then the DFA

looks like

q_{n_0} is the cycle start point



Since L is p-periodic we have for some sufficiently large $n_1 \in \mathbb{N}$

$$n \geq n_1 \implies \left(a^n \in L \iff a^{n+p} \in L \right)$$

It follows that if $n_0 + rp' \geq n_1$

we have

$$a^{n_0+rp'} \in L \iff a^{n_0+rp'+p} \in L.$$

$$\iff a^{n_0+rp'+2p} \in L$$

$$\iff a^{n_0+rp'+3p} \in L$$

$$\iff \cdots$$

It follows that

$$q_{n_0}, q_{n_0+p}, q_{n_0+2p}, \dots, q_{n_0+p'}$$

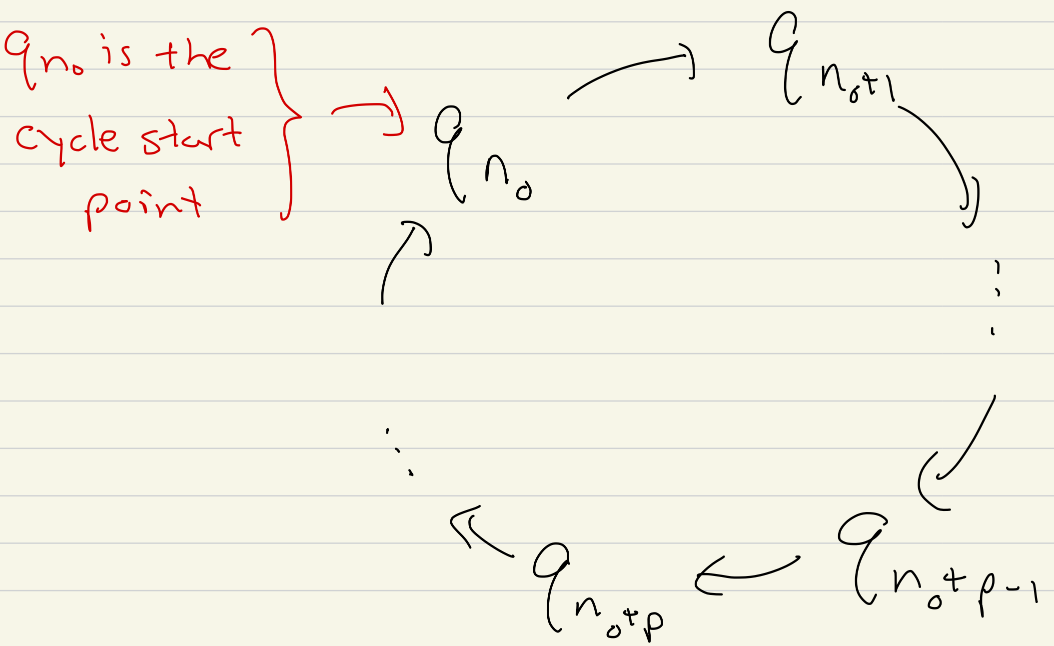
are either all accepting or all

rejecting.

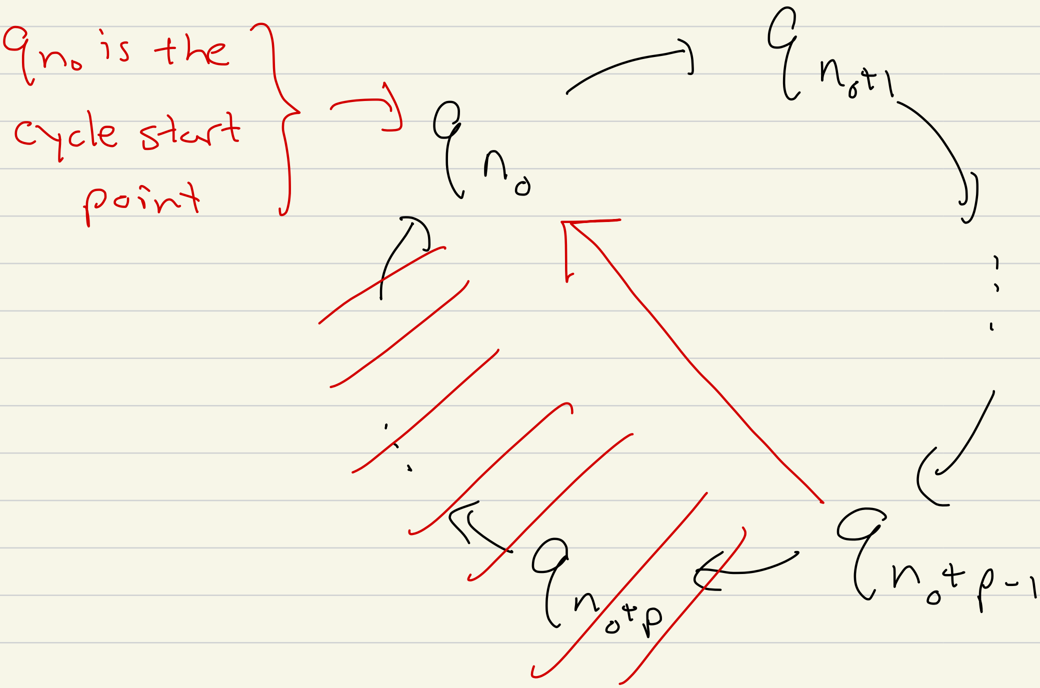
Similarly for $i = 0, 1, \ldots, p-1$

$q_{n_0+i}, q_{n_0+i+p}, q_{n_0+i+2p}, \ldots, q_{n_0+i+p'}$

are either all accepting or all rejecting. Hence we can replace



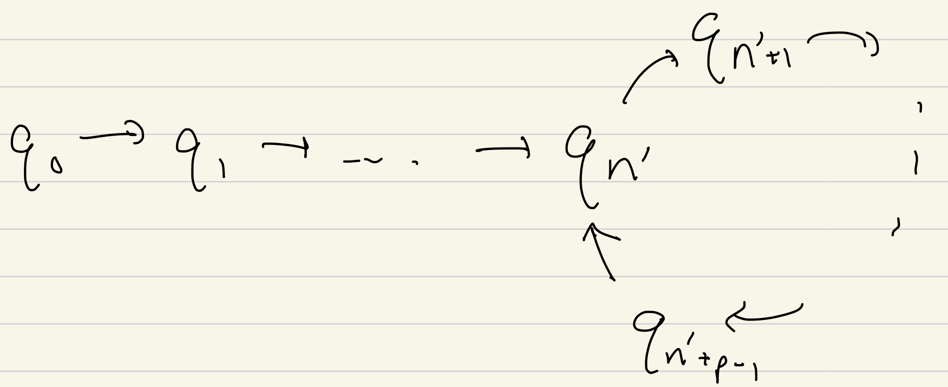
with the shorter cyclic part



eliminating $q_{n_0+p}, \ldots, q_{n_0+p'-1}$.

So if $p' > p$, this gives us a DFA with fewer states.

6.1.2(e) By Theorem 1.4, there exists a DFA with n_0+p states. By part (d) the smallest DFA recognizing L has cycle p, and some path length n, and has $n+p$ states



if $n' \leq n_0-1$, then running

the DFA on a^{n_0-1} lands in the cycle of the DFA, and so

$$a^{n_0-1} \in L \iff a^{n_0-1+p} \in L.$$

But in this case, not only does

$$a^n \in L \iff a^{n+p} \in L$$

hold for all $n \geq n_0$, but it also holds for all $n \geq n_0 - 1$.

Hence n_0 is <u>not</u> the smallest integer for which (1) holds.

(4) 6.1.3 : (a)

Since L is finite, we have

$a^n \notin L$ for n sufficiently large,

and so $\quad a^n \in L \iff a^{n+1} \in L$

for n sufficiently large (i.e. both

$a^n, a^{n+1} \notin L$). Hence L has

eventual period 1.

(b) We have $\quad a^n \notin L$ for $n \geq m+1$,

so

$$a^n \in L \iff a^{n+1} \in L \text{ for } n \geq m+1$$

Since $a^m \in L$ and $a^{m+1} \notin L$

we do <u>not</u> have

$$a^n \in L \iff a^{n+1} \in L$$

for $n = m$. Hence the smallest

n_0 satisfying equation (1) of

the handout is $n_0 = m+1$.

Hence by Exercise 6.1.2, the

DFA with the fewest states

accepting L has $n_0 + p =$

$(m+1)+1 = m+2$ states.

5 (a) $a^{24} = (a^5)^2 (a^7)^2$

$$a^{25} = (a^5)^5 (a^7)^0$$

$$a^{26} = (a^5)^1 (a^7)^3$$

$$a^{27} = (a^5)^4 (a^7)^1$$

$$a^{28} = (a^5)^0 (a^7)^4$$

(b) For $n \geq 29$ there is a $k \in \mathbb{N}$ such that

$$n - 5k \in \{24, 25, \ldots, 28\}$$

hence

$$a^n = (a^5)^k a^{n'}$$

where $n' \in \{24, \dots, 28\}$.

Hence, since $a^{n'}$ equals a power of a^5 times a power of a^7, the same is true for a^n.

(c) Say that $a^{23} = (a^5)^x (a^7)^y$.

Then $5x + 7y = 23$. Then and $x, y \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \dots\}$. Then

$$(5x + 7y) \bmod 7 = 23 \bmod 7$$

so

$$5(x \bmod 7) = 2.$$

But $0, 5, 10, 15, 20, 25 \bmod 7$

equal $0, 5, 3, 1, 6, 4$.

Hence $5(x \bmod 7) = 2$

implies that $(x \bmod 7) \geq 6$

so $x \geq 6$ so $5x + 7y \geq 30$

and hence $5x + 7y$ can't

equal 23.

(d) Since $a^n \in L$ for $n \geq 24$,

we have

$$a^n \in L \Leftrightarrow a^{n+1} \in L \text{ for } n \geq 24$$

so L has eventual period 1.

(e) We have $a^{23} \notin L$ and

$a^{24} \notin L$ have

$$a^n \in L \Leftrightarrow a^{n+1} \in L$$

is true for $n \geq 24$ but not $n \geq 23$.

Hence n_0 in Exercise 6.1.2

is 24, and $p = 1$, so the

Smallest number of states in a DFA recognizing L is

$$n_0 + p = 24 + 1 = 25.$$

[Note: By Exercise 6.1.3 of the handout $\{a\}^* \setminus L$ is a finite language whose longest string is a^{23}. Hence the fewest states in a DFA recognizing $\{a\}^* \setminus L$ is $23 + 2 = 25$.

Since the fewest states in a DFA for L is the same as for $\{a\}^* \setminus L$ (by reversing accepting and rejecting states), the fewest states in a DFA recognizing L is 25.

This gives another proof of the main result of this exercise.]