Homework 7 Solutions   CPSC 421/501 2024

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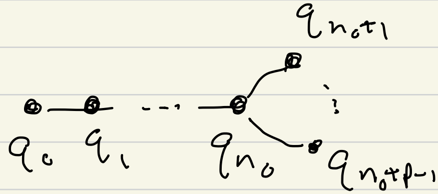
(2) Since L is regular, it has an eventual period $p \in \mathbb{N}$, and is recognized by a DFA



So if $n_i \geq n_0$ and $a^{n_i} \in L$ we have that

$$a^{n_i} \in L \iff a^{n_i+p} \in L$$

and hence $a^{n_i+p} \in L$ and so $n_{i+1} \leq n_i + p$.

Since $n_1 < n_2 < n_3 < \cdots$, there are only finitely many i such that $n_i < n_0$.

Letting i' be the largest integer

such that $n_{i'} < n_0$, it follows that

$$i > i' \implies n_{i+1} \leq n_i + p.$$

Setting

$$p' = \max(n_2 - n_1, n_3 - n_2, \dots, n_{i'+1} - n_{i'}),$$

it follows that (p' is finite, i.e. $p' \in \mathbb{N}$) and for any i,

$$n_{i+1} \leq n_i + \max(p, p').$$

(3) (a) Say that

$$n_1 = 3, \; n_3 = 6, \; n_5 = 9, \; n_7 = 12, \dots$$

Then $3 < n_2 < 6$ so $n_2 =$ either 4 or 5

and $6 < n_4 < 9$ so $n_4 =$ either 7 or 8

$\vdots$

and, for any $j \in \mathbb{N}$

$$n_{2j} = \text{either } 3j+1 \text{ or } 3j+2.$$

To each $S \subset \mathbb{N}$, we can therefore satisfy

the above, taking

$$n_{2j} = \begin{cases} 3j+1 & \text{if } j \in S \\ 3j+2 & \text{if } j \notin S \end{cases}$$

This gives a map

$$f : \text{Power}(\mathbb{N}) \to \{ \text{sequences } n_1 < n_2 < \dots \}$$

which is injective (since if $S, S' \in \mathrm{Power}(\mathbb{N})$ and $S \neq S'$, then for some $j \in \mathbb{N}$ either $j \in S$ and $j \notin S'$ or vice versa, and so n_{2j} meaning $n_{2j}(S)$ and $n_{2j}(S')$ are different). Hence f is injective. Since $\mathrm{Power}(\mathbb{N})$ is uncountable, so is the image of f.

Since $n_{2j} = 3j+1, 3j+2$ and $n_{2j-1} = 3j$ and $n_{2j+1} = 3j+3$, we have

$$\forall i \in \mathbb{N} \quad n_{i+1} - n_i = 1 \text{ or } 2.$$

[There are many other injections

$$f : \mathrm{Power}(\mathbb{N}) \longrightarrow \{\text{sequences } n_1 < n_2 < \ldots\}$$

that one can give, such as

$$f(S) = \{ 2i \mid i \in S \} \cup \{1,3,5,7,\ldots\}.\ ]$$

(b) We claim that the set of regular languages is countably infinite; here is a proof.

Each DFA for a language over $\{a\}$ is characterized by $n_0 \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{N}$ and the subset $F \subset Q = \{q_0, q_1, \ldots, q_{n_0+p-1}\}$ of accepting states. Since the names of the states of Q are unimportant, for each p, n_0 there are finitely many regular languages, L, recognized by a language with $p+n_0$ states. So we may list all languages in phases, namely Phase 1, Phase 2, ... where Phase k, for $k=1,2,\ldots$, is the set of languages with $p+n_0=k$ (for each k, $p+n_0=k$ implies that $n_0=0,1,\dots,k-1$ and $p=k-n_0$, so there are finitely many pairs (n_0,p) with $n_0 \in \mathbb{Z}_{\geq 0}$, $p \in \mathbb{N}$, and $n_0+p=k$). Hence Phase k lists only finitely many regular languages, and each regular language appears in at least one of Phase k for some $k \in \mathbb{N}$. Hence the set of regular languages is countably infinite.

Since there are countably many regular languages and uncountably many languages of the form $L_{n_1,n_2,\dots}$, some language of the form $L_{n_1,n_2,\dots}$ is uncountable.

[One can alternative describe such an L, somewhat explicitly, by forming a list

$(m_0,p_0), (m_1,p_1), (m_2,p_2), \ldots$

such that each pair $(m,p) \in \mathbb{Z}_{\geq 0} \times \mathbb{N}$ appears as (m_i,p_i) with $i \geq n+p$

(you can simply list all such pairs in and repeat some pairs if need be; here you don't actually need repition).

Then set $a^i \in L$ iff $a^{i-m_i} \notin L$.

It follows that L is not recognized by a DFA with path length m_i and cycle length p_i.]

(4) For $m \geq 1$ we have that whether or not $a^m \in L$ depends only on $m \bmod 12$, since the same is true for both

$$\{a^{3n} \mid n \in \mathbb{N}\} \text{ and for } \{a^{4n} \mid n \in \mathbb{N}\}.$$

Specifically:

$$m \geq 1 \implies \quad a^m \in L \text{ iff}$$
$$m \bmod 12 = 0, 3, 4, 6, 8, 9$$

So L is 12-eventually periodic. Hence the eventual period, p, of L divides 12.

If $p < 12$, then p must be a divisor of 12, $p = 1, 2, 3, 4, 6$.

<u>L cannot be eventually 6-periodic</u>

we have for all $k \in \mathbb{N}$

$$a^{12k+4} \in L \text{ but } a^{(12k+4)+6} \notin L$$

<u>L cannot be eventually 4-periodic</u>:

for all $k \in \mathbb{N}$ we have

$$a^{12k+6} \in L \text{ but } a^{(12k+6)+4} \notin L.$$

Hence $p \neq 1,2,3,6$, since L isn't eventually 6 periodic, and $p \neq 4$ since L isn't eventually 4 periodic.

Hence $p = 12$, i.e. 12 is the eventual period of L.

(5) $\mathrm{AccFut}_L(\varepsilon) = \{a^3\}$

$\quad\;\; \text{"} \quad (a) = \{a^2\}$

$\quad\;\; \text{"} \quad (a^2) = \{a\}$

$\quad\;\; \text{"} \quad (a^3) = \{\varepsilon\}$

$\quad\;\; \text{"} \quad (a^n) = \emptyset$ for $n \geq 4$.

Hence the smallest DFA for L has 5 states

$$\begin{tikzcd}
\arrow[r] & q_0 \arrow[r, "a"] & q_1 \arrow[r, "a"] & q_2 \arrow[r, "a"] & q_3 \arrow[r, "a"] & q_4 \arrow[loop right, "a"] \\
& \mathrm{AccFut} = \{a^3\} \arrow[u] & \mathrm{AccFut} = \{a^2\} \arrow[u] & \mathrm{AccFut} = \{a\} \arrow[u] & \mathrm{AccFut} = \{\varepsilon\} \arrow[u] & \mathrm{AccFut} = \emptyset \arrow[u]
\end{tikzcd}$$

with the correspondence

[Also a state is an accepting/final state iff ε lies in AccFut_L, so q_3 is the only accepting state.]