Group Homework 10.5, 2024, solutions

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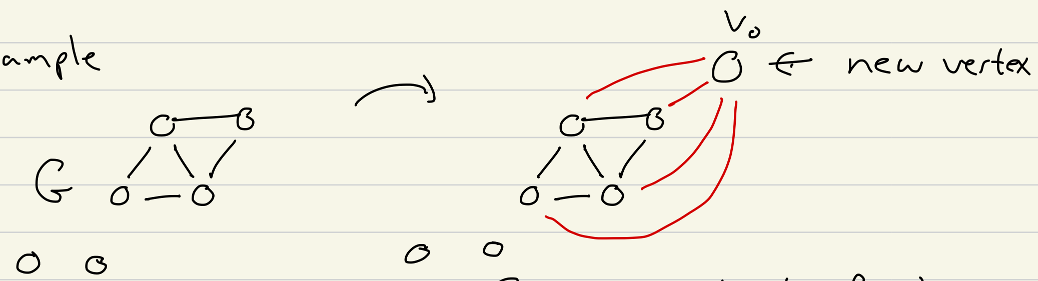
(2) 4COLOR $\in$ NP by non-determinstically guessing a 4-colouring of the graph (i.e. for each vertex connected to at least one edge).

To reduce 3COLOR to 4COLOR, given a graph G, add a new vertex, v_0, to G, connected to every vertex of G (connected to some edge)

Example



$G + v_0 +$ a bunch of edges

The new graph, G', can be described by having one more vertex, and at most 2 new edges for each edge of G. Hence $|\langle G' \rangle|$ can be generated from G in poly time.

Any 4-colouring of G' gives a 3-colouring of G with the 3 colours different than the colour of v_0; conversely, any 3-colouring of G gives a 4 colouring of G' by colouring v_0 with the $4^{\underline{th}}$ colour. Hence

$$\langle G \rangle \in 3\,\text{COLOR} \iff \langle G' \rangle \in 4\,\text{COLOR}.$$

So the map $G \mapsto G'$ gives a reduction $3\,\text{COLOR} \leq 4\,\text{COLOR}$.

Hence 4COLOR is NP-complete.

(3) DOUBLE-3-SAT is in NP by non-deterministically guessing two satisfying assignments for an input $\langle \varphi \rangle$, where φ is a 3CNF formula.

To show DOUBLE-3-SAT is NP-complete, it suffices to reduce 3SAT to DOUBLE-3-SAT.

If φ is in 3CNF, let $m \in \mathbb{N}$ be the smallest natural number such that φ does not contain the variable x_m.

Then $\varphi \wedge (x_m \vee x_m \vee \neg x_m)$ has 2 satifying assignments for each satisfying assignment of φ, namely where we take $x_m = T$ and $x_m = F$. Hence $\varphi \in 3SAT$ iff $\varphi \wedge (x_m \vee x_m \vee \neg x_m) \in$ DOUBLE-3-SAT.

Since we can find m by keeping track of all i with x_i occurring in φ, we can find m in polynomial time in $\langle \varphi \rangle$, and each of $x_1, \ldots, x_{m-1}$ must occur in φ. Hence (unless φ is empty),

$$\langle \varphi \rangle \geq \langle x_1 \rangle + \ldots + \langle x_{m-1} \rangle$$
$$\geq \sum_{i=1}^{m-1} (1 + \log_{10}(i+1)) \geq m-1 \quad \text{(crudely)},$$

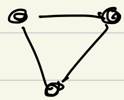
so adding the phrase

$$\wedge (x_m \vee x_m \vee \neg x_m)$$

to φ doesn't increase the length of the description by more than 9 symbols (of $($, $)$, $\wedge$, $\vee$, x) plus $3 \cdot \lceil \log_{10}(m+1) \rceil$, hence no more than polynomial in $\langle \varphi \rangle$. Hence $3SAT \leq_{poly}$ DOUBLE-3-SAT.

Hence DOUBLE-3-SAT is NP-complete

(4) (a) G_1 :  (one vertex, no edges)

G_2 :  three vertices, each two vertices joined by an edge

(b) We can go through all settings of the variables of a 3CNF formula φ (in, for example, lexicographical order, from $FF\ldots F$ to $TT\ldots T$) in space at most the number of variables, plus some space to evaluate φ at each of the settings. This tells us whether or not $\varphi \in 3SAT$; if it is, we write $\langle G_1 \rangle$, if not, we write $\langle G_2 \rangle$.

(c) If $P = NP$, then there is a polynomial time algorithm to decide if a 3CNF φ is satisfiable or not; by writing $\langle G_1 \rangle$ if it is, and $\langle G_2 \rangle$ if it isn't, we get a polynomial time reduction $3SAT \leq_{poly} 2COLOUR$.

If $P \neq NP$, then 3SAT can't be poly time reducible to 2COLOUR, since $2COLOUR \in P$ and hence $3SAT \leq_{poly} 2COLOUR$ would imply $3SAT \in P$ and hence $P = NP$.

Therefore

$$P = NP \iff 3SAT \leq_{poly} 2COLOUR.$$

Since, as of November 2024, we don't know whether or not $P = NP$, we don't know whether or not $3SAT \leq_{poly} 2COLOUR$.

(5) Same as 4 part (b), except that instead of writing $\langle G_1 \rangle$ or $\langle G_2 \rangle$ we, respectively, accept or reject.

(6) If $L \in PSPACE$ then there is a poly-space deterministic algorithm to decide L. By exchanging "accept" and "reject," we have L^{comp} (the complement of L) $\in PSPACE$.

If $L \in NPSPACE$, then by Savitch's Theorem, $L \in PSPACE$, and hence $L^{comp} \in PSPACE$. Since $PSPACE \subset NPSPACE$, $L^{comp} \in NPSPACE$.