CPSC 421/501 Sept 4, 2024

Today:

- Admin
- Start Cantor's Theorem

Instructor: Joel Friedman

Course website:

https://www.cs.ubc.ca/~jf/courses/421.F2024/index.html

"In mathematics, you don't understand things. You just get used to them."

– John von Neumann

["In mathematics, it takes a while for things to 'sink in'."]

Admin:

Grade:

$$0.55\, f + 0.35 \max(f, m) + 0.10 \max(f, m, h)$$

f = final, m = midterm,

h = homework (drop lowest 3).

Homework: Due on

Thursdays, 11:59 pm,

gradescope

Midterm: Friday, Nov 1, in class time.
Location: To be announced

Group Homework: Groups of size ≤ 4. One submission per group.

Individual Homework: One submission per student; must be your own work.

$\boxed{\text{Some weeks we will not grade all homework parts.}}$

[Homework must be legible.]

First 2 weeks:

1. Show the "halting problem" is "unsolvable," meaning ...

... "undecidable" (but it is "recognizable")

2. Give you an idea of the typical level of difficulty in CPSC 421/501

3. Start with "Uncomputability in CPSC 421/501"

Cantor's Theorem:

- We work in "naive set theory"
- Preliminary definitions:
  - Power(S), S a set
  - Image(f)
  - "f is surjective"

  } f a function
- State Cantor's Theorem
- Give examples, formal proof, remarks

If S is a set, $\text{Power}(S)$, the <u>power set</u> of S is the set of all subsets of S.

$$\text{Power}(\{1,2,3\})$$

$$= \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$$

Rem: If $|S| = 3$ (i.e. S is of size/cardinality 3),

then

$$\underbrace{|\mathrm{Power}(S)| = 2^{|S|}}_{\text{holds for any finite set.}} = 8$$

Warning: Infinite sets can have "interesting properties" that aren't true for finite sets...

If A, B are sets, and

$f : A \to B$ is a function,

then

(1) $\text{Image}(f) = \{ f(a) \mid a \in A \}$

(a subset of B)

(2) We say that f is

surjective (onto, epimorphism)

if $\text{Image}(f) = B$

Example: $f : A \to B$,

$A = \{1, 2, 3\}$, $B = \{x, y, z\}$

given by

(1) $f(1) = x$, $f(2) = f(3) = y$

OR

(2) f maps $1 \mapsto x$, $2 \mapsto y$, $3 \mapsto y$

(3)

$$\begin{tikzcd} 1 \arrow[r] & x \\ 2 \arrow[r] & y \\ 3 \arrow[ur] & z \end{tikzcd}$$

$\mathrm{Image}(f) = \{x, y\}$

$$A \xrightarrow{f} B$$

Then

$$\text{Image}(f) = \{x, y\}$$

$$\neq \{x, y, z\} = B$$

So f is not surjective

Example 2: same A, B,

$$f: \quad 1 \longrightarrow x, \quad 2 \longrightarrow z, \quad 3 \longrightarrow y$$

$\text{Image}(f) = B$

surjection

Example:

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

(Zahlen)

$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$$

$$\mathbb{N} = \{1, 2, \ldots\} \overset{\text{proper}}{\subset}$$

$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$$

but $f: \mathbb{N} \to \mathbb{Z}_{\geq 0}$

given by $f(x) = x - 1$

is a surjection (bijection)

$$
\begin{array}{ccc}
\mathbb{N} & & \mathbb{Z}_{\geq 0} \\
1 & \longrightarrow & 0 \\
2 & \longrightarrow & 1 \\
3 & \longrightarrow & 2 \\
4 & \longrightarrow & 3 \\
\vdots & &
\end{array}
$$

Remark: If $S \overset{\text{proper}}{\subset} T$,

S is a subset of T, and

$S \neq T$, and T is finite

$$\begin{array}{ccc} S & & T \\ \{x, z\} & \xrightarrow{f} & \{x, y, z\} \end{array}$$

There is no surjection $f: S \to T$

Theorem: [Cantor] Let S be a set, and

$$f : S \to \mathrm{Power}(S)$$

(e.g. $S = \{x, y\}$,

$$S \xrightarrow{f} \mathrm{Power}(S)$$

$$\begin{array}{lcl} x & \longrightarrow ? & \emptyset \\ y & \longrightarrow ? & \{x\} \\ & & \{y\} \\ & & \{x, y\} \end{array}$$

Then

$$T = \{ s \mid s \notin f(s) \}$$

is not in the image of f.

E.g.

$$\begin{tikzcd}
1 \arrow[r, "f"] & \emptyset & \mathrm{Image}(f) \arrow[l] \arrow[dddl, bend left] \\
 & \{1\} & T \arrow[l] \\
2 \arrow[dr] & \{2\} & \\
 & \{1,2\} &
\end{tikzcd}$$

So! $f(1) = \emptyset$,

$1 \notin f(1)$

so T contains 1;

$f(2) = \{1, 2\}$, so $2 \in f(2)$

so T doesn't contain 2

$$T = \{1\}$$