

CPSC 421/501

Sept 4, 2024

Today:

- Admin

- Start Cantor's Theorem

Instructor: Joel Friedman

Course website:

<https://www.cs.ubc.ca/>

[~jif/courses/421.F2024/](https://www.cs.ubc.ca/~jif/courses/421.F2024/)

[index.html](https://www.cs.ubc.ca/~jif/courses/421.F2024/index.html)

"In mathematics, you don't understand things. You just get used to them."

- John von Neumann

"In mathematics, it takes a while for things to 'sink in'."

Admin:

Grade:

$$0.55 f + 0.35 \max(f, m) \\ + 0.10 \max(f, m, h)$$

f = final, m = midterm,

h = homework (drop lowest 3).

Homework: Due on

Thursdays, 11:59 pm,

gradescope

Midterm! Friday, Nov 1,
in class time.

Location! To be announced

Group Homework! Groups
of size ≤ 4 , One
submission per group.

Individual Homework! One
submission per student;
must be your own work.

Some weeks we will not grade all homework parts.

[Homework must be legible.]

First 2 weeks!

(1) Show the "halting problem" is "unsolvable," meaning ...
... "undecidable" (but it is "recognizable")

(2) Give you an idea of the typical level of difficulty in CPSC 421/501

(3) Start with "Uncomputability in CPSC 421/501"

Cantor's Theorem:

- We work in "naive set theory"

- Preliminary definitions:

- $\text{Power}(S)$, S a set

- $\text{Image}(f)$

- " f is surjective"

} f a

function

- State Cantor's Theorem

- Give examples, formal proof, remarks

If S is a set, $\text{Power}(S)$,
the power set of S is
the set of all subsets of S .

$$\text{Power}(\{1, 2, 3\})$$

$$= \left\{ \emptyset, \{1\}, \{2\}, \{3\}, \right. \\ \left. \{1, 2\}, \{1, 3\}, \{2, 3\}, \right. \\ \left. \{1, 2, 3\} \right\}$$

Rem: If $|S| = 3$ (i.e. S
is of size/cardinality 3),
then

$$|\text{Power}(S)| = 2^{|S|} = 8$$

holds for any finite set.

Warning: Infinite sets can
have "interesting properties"

that aren't true for finite sets...

If A, B are sets, and

$f: A \rightarrow B$ is a function,

then

$$(1) \text{Image}(f) = \{ f(a) \mid a \in A \}$$

(a subset of B)

(2) We say that f is

surjective (onto, epimorphism)

if $\text{Image}(f) = B$

Example: $f: A \rightarrow B$,

$$A = \{1, 2, 3\}, \quad B = \{x, y, z\}$$

given by

$$(1) \quad f(1) = x, \quad f(2) = f(3) = y$$

OR

$$(2) \quad f \text{ maps } 1 \mapsto x, \quad 2 \mapsto y, \quad 3 \mapsto y$$

$$(3) \quad \begin{array}{ccc} 1 & \longrightarrow & x \\ 2 & \longrightarrow & y \\ 3 & \longrightarrow & z \end{array} \quad \left. \vphantom{\begin{array}{ccc} 1 & \longrightarrow & x \\ 2 & \longrightarrow & y \\ 3 & \longrightarrow & z \end{array}} \right\} \text{Image}(f) = \{x, y\}$$

$$A \xrightarrow{f} B$$

Then

$$\text{Image}(f) = \{x, y\}$$

$$\neq \{x, y, z\} = B$$

So f is not surjective

Example 2: same A, B ,

$$f: \begin{array}{ccc} 1 & \longrightarrow & x \\ 2 & & y \\ 3 & & z \end{array}$$

$$\begin{array}{ccc} 2 & \longrightarrow & y \\ 3 & \longrightarrow & z \end{array}$$

$$\text{Image}(f) = B$$

surjection

Example:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

(Zahlen)

$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$$

$$\mathbb{N} = \{1, 2, \dots\} \quad \begin{array}{l} \text{proper} \\ \subset \end{array}$$

$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$$

but $f: \mathbb{N} \rightarrow \mathbb{Z}_{\geq 0}$

given by $f(x) = x - 1$

is a surjection (bijection)

\mathbb{N} $\mathbb{Z}_{\geq 0}$

1 \longrightarrow 0

2 \longrightarrow 1

3 \longrightarrow 2

4 \longrightarrow 3

,

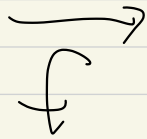
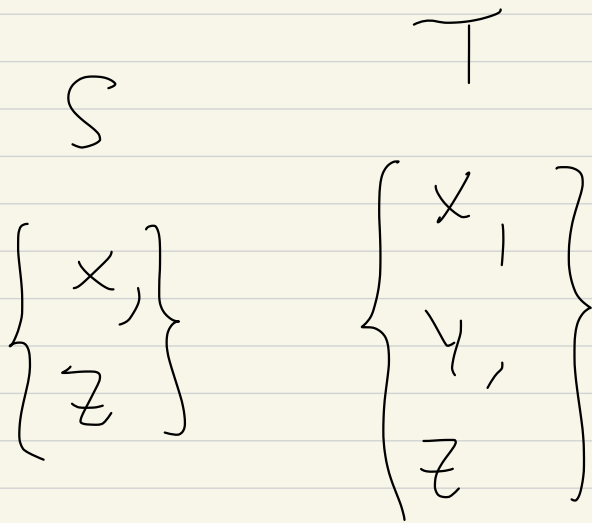
,

,

Remark: If $S \subset^{\text{proper}} T$,

S is a subset of T , and

$S \neq T$, and T is finite



There is no surjection $f: S \rightarrow T$

Theorem: [Cantor] Let

S be a set, and

$$f: S \rightarrow \text{Power}(S)$$

(eg. $S = \{x, y\}$,

$$S \xrightarrow{f} \text{Power}(S)$$

$$x \longrightarrow ? \quad \emptyset$$

$$y \longrightarrow ? \quad \{x\}$$

$$ \longrightarrow ? \quad \{y\}$$

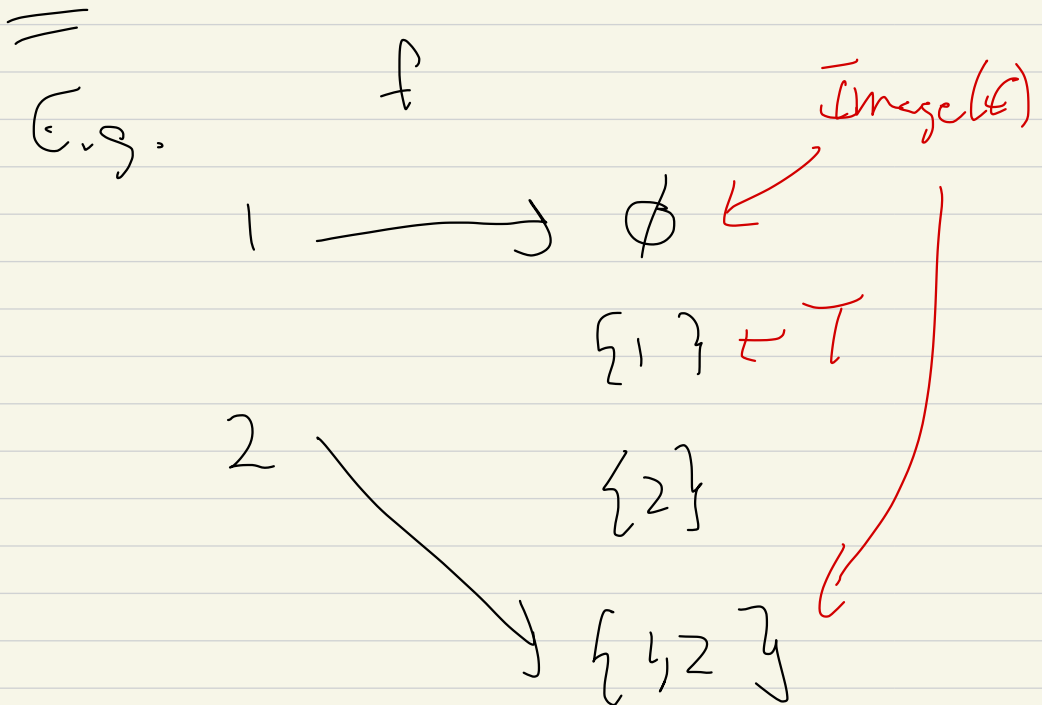
$$ \longrightarrow ? \quad \{x, y\}$$

Then

$$T = \{s \mid s \notin f(s)\}$$

is not in the image of f .

\equiv



So! $f(1) = \emptyset,$

$1 \notin f(1)$

so T contains 1;

$f(2) = \{1, 2\}$, so $2 \in f(2)$

so T doesn't contain 2

$$T = \{1\}$$