CPSC 421/501 Sept 6, 2024

Today:

- Cantor's Theorem
- Injections, surjections, bijections
- What does $|S| \le |B|$ mean?

  for (infinite) sets
- Generalized Cantor's Theorem
- Start: - Alphabets
  - Strings
  - Languages
  - Complexity Classes

    are ... ?

First 2 weeks:

Most of Sections 1–7

of

Uncomputability in CPSC 421/501

Last time:

Cantor's Theorem: Let

$f: S \to \text{Power}(S)$. Then

$$\boxed{T = \{ s \in S \mid s \notin f(s) \}}$$

is not the image.

hopefully today

Piazza & Office

will start next week

LaTeX example will

appear

Remark on Handout:

Uncomputability in CPSC 421/501

The exercises are undergoing many changes, including exercise numbers. But assigned homework numbers will not change.

Example 1: $S = \{1, 2\}$

say that f given by

$1 \longrightarrow \emptyset$ (empty set)

$\{1\} \leftarrow T$

2 $\quad \{2\}$

$2 \longrightarrow \{1, 2\} = S$

$S \qquad \text{Power}(S)$

$f(1) = \emptyset$

$f(2) = \{1, 2\}$

Another Example

$$\begin{array}{ccc}
 & & \emptyset \\
 & & \{1\} \\
1 & & \{2\} \\
2 & & \{3\} \\
3 & & \{1,2\} \\
 & & \vdots \\
S & & \{1,2,3\} \\
 & & \\
 & & \text{Pow}(S)
\end{array}$$

$T = \{ s \in S \mid s \notin f(s) \}$

In Example 1:

$1, \quad f(1) = \emptyset$

$1 \notin \emptyset$

so $\boxed{\begin{array}{l} 1 \notin f(1) \\ 1 \in T \end{array}}$

$\left(\text{if } u \in T \text{ but } u \notin \underset{f(1)}{T'} \right.$

are different $\Big)$

$f(1) \neq T$

Similarly

$$2 \in f(2) = \{1,2\}$$

so $2 \in f(2)$, $2 \notin T$

$$f(2) \neq T.$$

Example 3 (K.T.'s example)

$$\begin{array}{ccc} & \emptyset & \emptyset \\ 1 & \longrightarrow & \{1\} \\ & & \{2\} \\ 2 & \longrightarrow & \{1,2\} \end{array}$$

$$T = \emptyset$$

$$T = \{ s \in S \mid s \notin f(s) \}$$

$$1 \in f(1) \Rightarrow 1 \notin T$$

$$2 \in f(2) \Rightarrow 2 \notin T$$

$$T = \emptyset$$

Example 4: (D.J. example)

$$2 \mapsto \{2\}$$

$$1 \mapsto \{1\}$$

$$1 \in f(1),\ 2 \in f(2) \Rightarrow T = \emptyset$$

Looking ahead ... ultimately

set $S = \Sigma^*$

maybe Σ_{ASCII}^*

Proof of Cantor's Theorem:

$f: S \to Power(S)$,

$T = \{ s \in S \mid s \notin f(s) \}$

Say that $T \in Image(f)$

then $T = f(t)$

Either $t \in f(t) = T$

$t \notin f(t) = T$

We'll show both cases are impossible.

$\boxed{t \in f(t)} = T = \{s \mid s \notin f(s)\}$

so $t \in T$ then $t = s$, $s \notin f(s)$

$\boxed{t \notin f(t)}$

impossible

Similarly, if

$t \notin f(t) = T = \{s \mid s \notin f(s)\}$

$t \notin \{s \mid \underbrace{s \notin f(s)}_{\text{this isn't true when } t = s}\}$

$s \in f(s)$

when $s = t$

Alt proof

(A.f.)

$t \notin f(\cancel{t}) = T$

$t \notin T$

by def of T

$t \in f(t)$

$t \in f(t) = T$

$S \to \text{Power}(S)$

$$\boxed{\text{Say } |S| \leq |B|}$$

$f: S \to \text{Power}(S)$

$\leq$

all f not surjective

$\text{Power}(B)$

B

S

	Are you ≥ 3 metres tall	Have you met one of your grandchildren	Have you seen the TV series Dark
Student 1	no		
Student 2		no	
Student 3			no
Fictive Student	yes	yes	yes

$$S \xrightarrow{f} \mathrm{Power}(B)$$

$f(\text{student } 1)$

$= \{\text{no}, \text{no}, \text{no}\} = \emptyset$ in B

$f(\text{student } 2) = \{\text{no}, \text{no}, \text{no}\}$

$= \emptyset$ in B

$f(\text{student } 3) = \emptyset \in B$

Answers

student 1 $\{no, no, no\}$

student 2 $\{yes, no, no\}$

student 3 $\{yes, no, yes\}$

$\vdots$

Map $B \to \{no, yes\}$

View as a subset of B

$$\begin{tikzcd}
 & \emptyset \\
1 \arrow[ru] & \{1\} \\
2 \arrow[rd] & \{2\} \\
 & \{1,2\} \\
S & \mathrm{Pow}(S)
\end{tikzcd}$$

$f(1) = \emptyset$ $\qquad$ $f(2) = S$

Is $x \in f(y)$	$y = 1$	$y = 2$
$x = 1$	no	
$x = 2$		yes

$\qquad\qquad\qquad\downarrow \qquad\qquad \downarrow$

$\qquad\qquad\qquad$ yes $\qquad\quad$ no

$$T = \{1 \qquad\qquad\qquad \}$$