

Today?

- $|S| \leq |B|$, $|S| = |B|$, $|S| < |B|$
- Generalized Cantor's Theorem
- yes/no view
- strings

Last time?

$$S \rightarrow \text{Power}(S) \text{ can}$$

never be surjective. Usually:

$$S \rightarrow \text{Power}(B)$$

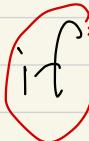
where B is "bigger" or

"at least as large as" S

=

Def:

We write, for sets, $S \leq B$,

$|S| \leq |B|$ if there is  if

an injection $S \rightarrow B$.

=

We write $|S| = |B|$ if there is a bijection $S \rightarrow B$.

=

We write $|S| < |B|$ if

$|S| \leq |B|$ and $|S| \neq |B|$.

For finite sets,

$|S|, |B|$ the size of $S,$

the size of $B, 0, 1, 2, \dots$

\equiv

$|S|=2, S=\{1, 2\}$

$|B|=3, B=\{1, 2, 3\}$

(here $S \subset B$)

1 → 1

2 → 2

3

S B

So $S \subset B$, $|S| \leq |B|$

Example:

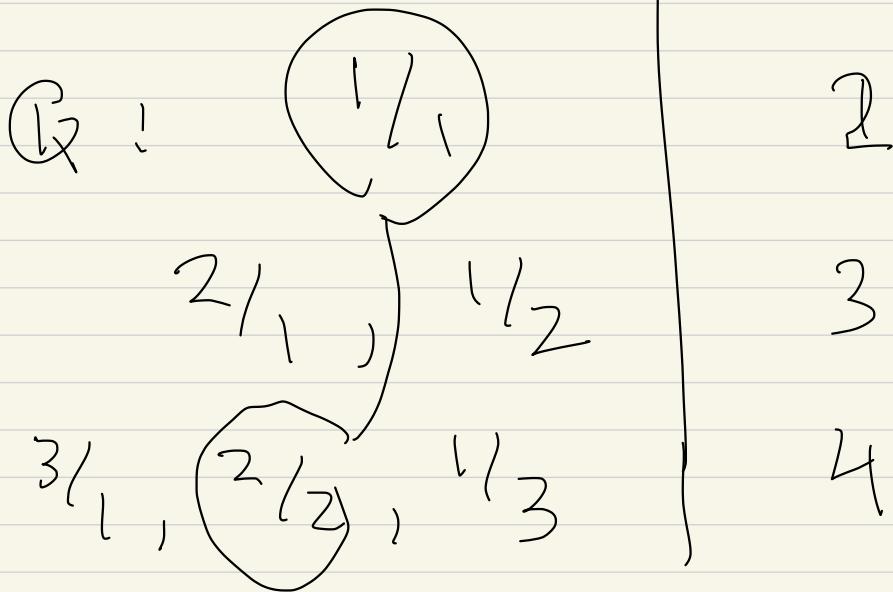
$\mathbb{N} = \{1, 2, 3, \dots\}$

\mathbb{Q}^+ = positive rational

numbers = $\left\{ \frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \dots \right\}$

$$|\mathbb{Q}^+| \leq |\mathbb{N}|$$

num + denon



So $1/1 \quad 2/1 \quad 3/1$ ✓

$1/2 \quad 2/2$

$1/3$

$\mathbb{Q}^+ : 1/1, 2/1, 1/2,$

$3/1, \cancel{1/2}, 1/3,$

$4/1, 3/2, 2/3, 1/4, \dots$

Then

$\mathbb{Q}^+ : 1/1, 2/1, 1/2, 3/1, 1/3, \dots$

f $\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

IN: 1 2 3 4 5 ..

$\downarrow = \text{"maps to"}$

Gives a bijection
(injection)

$$f: \mathbb{Q}^+ \rightarrow \mathbb{N}$$

Def We say that S is countably infinite if there is a bijection $S \rightarrow \mathbb{N}$

$$S \rightarrow \mathbb{N}$$

Say Σ is an alphabet if

Σ is a finite non-empty set.

For $k = 0, 1, 2, \dots$

Σ^k = "set of strings of length

k over Σ "

$$= \underbrace{\Sigma \times \Sigma \times \dots \times \Sigma}_{k \text{ copies}}$$

e.g. $\Sigma = \{a, b\}$

$$\Sigma^0 = \{\epsilon\}$$

ϵ = Empty string

$$\Sigma^1 = \{(a), (b)\}$$

$$\Sigma^2 = \{(a,a), (a,b), (b,a), (b,b)\}$$

$$= \{aa, ab, ba, bb\}$$

Set of all strings over Σ

$$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \dots$$

E.g. $\Sigma = \{a, b\}$

$$\Sigma^* = \{ \epsilon, a, b, aa, ab, ba, bb, aaaa, \\ aabb, \dots \}$$

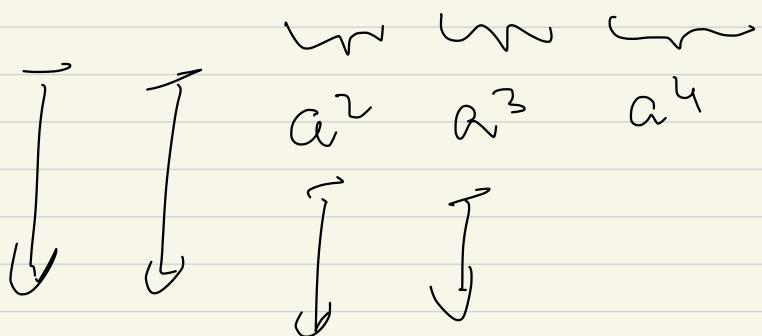
(typically list Σ^* , or a subset thereof, in order of increasing length, within that in lexicographical (dictionary) order.

Claim! For any alphabet, Σ ,

Σ^* is countably infinite.

e.g. $\Sigma = \{a\}$

$\Sigma^* = \{ \epsilon, a, aa, aaa, aaaa, \dots \}$



1, 2, 3, 4, ...

e.g. $\Sigma = \{a, b\}$

Length
0 ϵ = Empty string
 $\neq \emptyset$

Length
1 a → 2
b → 3

a a → 4
a b → 5
b a → 6
b b → 7

1
2
3

What is

$\Sigma = \{A, B, C\}$ and Σ

\sum

$$\{\{1, 2\}\} \rightarrow \sum \quad \left. \begin{array}{l} |\Sigma^2| = \\ |\Sigma|^2 \end{array} \right\} |\sum| = |\Sigma|$$

$$\emptyset \rightarrow \sum \quad \left. \begin{array}{l} |\Sigma^0| = 1 \end{array} \right\}$$

\sum

$$\Sigma = \{\sum\}$$

\downarrow

$$(\emptyset \rightarrow \sum)$$

$$1 \rightarrow \text{elt of } \Sigma$$

$$2 \rightarrow \text{elt of } \Sigma$$

$$(a, b) = ab$$

$$1 \mapsto a$$

$$2 \mapsto \underline{b}$$

$$(a, b, b, a) = abba$$

$$1 \mapsto a$$

$$2 \mapsto b$$

$$3 \mapsto b$$

$$4 \mapsto a$$

$$\Sigma = \{a, b, c, d\}$$

aabbdc

1	\mapsto	a
2	\mapsto	a
3		b
4		b
5		d
6		c

$$\Sigma = \{c, b\}$$

N

$$\Sigma \hookrightarrow 1$$
$$a \hookrightarrow 2$$
$$b \hookrightarrow 3$$
$$ac \hookrightarrow 4$$
$$ab \hookrightarrow 5$$

ba

bb

This idea gives a bijection

$$\Sigma^* \rightarrow \mathbb{N}$$

therefore injection

$$\Sigma^* \rightarrow \mathbb{N}$$

So

$$|\Sigma^*| \leq |\mathbb{N}|$$

$$|\Sigma^*| = |\mathbb{N}|$$

$$S \xrightarrow{\text{inj}} \text{Power}(S)$$

$$s \rightarrow \{s\} \in \text{Power}(S)$$

Cantor's Theorem: There is
no surjection

$$S \rightarrow \text{Power}(S)$$

Hence: $|S| \neq |\text{Power}(S)|$

So $|S| < |\text{Power}(S)|$

$$\sum k$$

$$\stackrel{\Leftarrow}{=} \sum x \dots x \sum$$

$\brace{k \text{ copies}}$

$$\stackrel{\Leftarrow}{=} \left\{ \begin{array}{l} 1 \rightarrow \sum \\ 2 \rightarrow \sum \\ \vdots \\ k \rightarrow \sum \end{array} \right\}$$

$$\stackrel{\Leftarrow}{=} \text{map } [k] \rightarrow \sum$$

$$[k] = \{1, \dots, k\}$$

Cor: Power ($\mathbb{N}^\mathbb{N}$)

Power (\sum^*), alphabet
 \sum

are uncountable.

$$\sum^* = \sum^0 \cup \sum^1 \cup \sum^2 \cup \dots$$

$$\sum \{ a, b \}$$

queue

$$1 \rightarrow a$$

$$2 \rightarrow a$$

$$3 \rightarrow a$$

$$4 \rightarrow a$$

$$5 \rightarrow a$$

$$\sum^k = \left\{ \text{mappings } \{1, \dots, k\} \right\}$$

to \sum

Generalized Cantor's Theorem:

Injective form:

Let $h: S \rightarrow B$ be an

injection, and

$f: S \rightarrow \text{Power}(B)$.

Then

$$T = \left\{ h(s) \mid s \in S \text{ and } h(s) \notin f(s) \right\}$$

is not in the image of f .