CPSC 421/501 Sept 9, 2024

Today:

- $|S| \le |B|$, $|S| = |B|$, $|S| < |B|$
- Generalized Cantor's Theorem
- yes/no view
- strings

Last time:

$S \to \text{Power}(S)$ can never be surjective. Usually:

$$S \to \text{Power}(B)$$

where B is "bigger" or "at least as large as" S

=

<u>Def</u>:

We write, for sets, S, B,

$|S| \leq |B|$ if there is an injection $S \to B$. iff

=

We write $|S| = |B|$ if there is a bijection $S \to B$.

=

We write $|S| < |B|$ if

$|S| \leq |B|$ and $|S| \neq |B|$.

For finite sets,

$|S|$, $|B|$ the size of S,

the size of B, $0, 1, 2, \ldots$

$=$

$|S| = 2$, $S = \{1, 2\}$

$|B| = 3$, $B = \{1, 2, 3\}$

(here $S \subset B$)

$$
\begin{array}{ccc}
1 & \longrightarrow & 1 \\
2 & \longrightarrow & 2 \\
 & & 3 \\
S & & B
\end{array}
$$

So $S \subset B$, $|S| \leq |B|$

Example:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$\mathbb{Q}^+$ = positive rational numbers $= \left\{\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \dots\right\}$

$|\mathbb{Q}^+| \le |\mathbb{N}|$

num + denom

$\mathbb{Q}$: $1/1$ — 2

$2/1$, $1/2$ — 3

$3/1$, $2/2$, $1/3$ — 4

So

$$
\begin{array}{ccc}
1/1 & 2/1 & 3/1 \\
1/2 & 2/2 & \\
1/3 & & \ddots
\end{array}
$$

$\mathbb{Q}^+$: $1/1$, $2/1$, $1/2$, $3/1$, $\cancel{2/2}$, $1/3$, $4/1$, $3/2$, $2/3$, $1/4$, ...

Then

$$
\begin{array}{cccccccc}
\mathbb{Q}^+ : & 1/1, & 2/1, & 1/2, & 3/1, & 1/3, & \dots \\
f & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\mathbb{N} : & 1 & 2 & 3 & 4 & 5 & \dots
\end{array}
$$

$\downarrow$ = "maps to"

Gives a bijection

(injection)

$$f: \mathbb{Q}^+ \to \mathbb{N}$$

Def We say that S is <u>countably infinite</u> if there is a bijection

$$S \to \mathbb{N}$$

Say Σ is an alphabet if

Σ is a finite non-empty set.

For $k = 0, 1, 2, \ldots$

Σ^k = "set of strings of length k over Σ"

$$= \underbrace{\Sigma \times \Sigma \times \ldots \times \Sigma}_{k \text{ copies}}$$

e.g. $\Sigma = \{a, b\}$

$\Sigma^0 = \{\varepsilon\}$ ε : Empty string

$\Sigma^1 = \{(a), (b)\}$

$\Sigma^2 = \{(a,a), (a,b), (b,a), (b,b)\}$

$= \{aa, ab, ba, bb\}$

Set of all strings over Σ

$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots$

e.g. $\Sigma = \{a, b\}$

$$\Sigma^* = \{ \varepsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots \}$$

(typically list Σ^*, or a subset thereof, in order of increasing length, within that in lexicographical (dictionary) order.

Claim: For any alphabet, Σ, Σ^* is countably infinite.

e.g. $\Sigma = \{a\}$

$$\Sigma^* = \{\varepsilon, a, \underbrace{aa}_{a^2}, \underbrace{aaa}_{a^3}, \underbrace{aaaa}_{a^4}, \ldots$$

$$\begin{array}{ccccc} \varepsilon & a & a^2 & a^3 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ 1, & 2, & 3, & 4, & \ldots \end{array}$$

e.g., $\Sigma = \{a, b\}$

Length 0: ε = empty string $\longmapsto$ 1

$\neq \emptyset$

Length 1:

$a \longmapsto 2$

$b \longmapsto 3$

$aa \longmapsto 4$

$ab \quad 5$

$ba \quad 6$

$bb \quad 7$

$\vdots$

What is $\Sigma^{i} = \{x_1 x_2 \ldots x_i\}$, and Σ^{0}

$\{1, 2\} \to \Sigma$

$\{1\} \to \Sigma$

$\emptyset \to \Sigma$

$\Sigma^{0} = \{\varepsilon\}$

$\downarrow$

$(\emptyset \to \Sigma)$

$|\Sigma^{2}| = |\Sigma|^{2}$

$|\Sigma^{1}| = |\Sigma|$

$|\Sigma^{0}| = 1$

$1 \to$ elt of Σ

$2 \to$ elt of Σ

$(a, b) = ab$

$1 \longmapsto a$

$2 \longmapsto b$

$(a, b, b, a) = abba$

$1 \longmapsto a$

$2 \longmapsto b$

$3 \longmapsto b$

$4 \longmapsto a$

$\Sigma = \{a, b, c, d\}$

aabbdc

$1 \longmapsto a$
$2 \longmapsto a$
$3 \quad b$
$4 \quad b$
$5 \quad d$
$6 \quad c$

$\Sigma = \{a, b\}$ $\qquad\qquad$ $\mathbb{N}$

$\varepsilon \longleftrightarrow 1$

$a \longleftrightarrow 2$

$b \longleftrightarrow 3$

$aa \longleftrightarrow 4$

$ab \longleftrightarrow 5$

ba

bb

This idea gives a bijection

$$\Sigma^* \longrightarrow \mathbb{N}$$

therefore injection

$$\Sigma^* \longrightarrow \mathbb{N}$$

so $$|\Sigma^*| \leq |\mathbb{N}|$$

$$|\Sigma^*| = |\mathbb{N}|$$

$$S \xrightarrow{\text{inj}} \text{Power}(S)$$

$$s \longrightarrow \{s\} \in \text{Power}(S)$$

Cantor's Theorem: There is no surjection

$$S \to \text{Power}(S)$$

Hence: $|S| \neq |\text{Power}(S)|$

so $|S| < |\text{Power}(S)|$

$$\Sigma^k$$

$$= \underbrace{\Sigma \times \dots \times \Sigma}_{k \text{ copies}}$$

$$\simeq \left\{ \begin{array}{l} 1 \to \Sigma \\ 2 \to \Sigma \\ \vdots \\ k \to \Sigma \end{array} \right\}$$

$$\simeq \text{map } [k] \to \Sigma$$

$$[k] = \{1, \dots, k\}$$

Cor: $\mathrm{Power}(\mathbb{N})$

$\mathrm{Power}(\Sigma^*)$, alphabet Σ

are <u>uncountable</u>.

$$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots$$

$\Sigma = \{a, b\}$

aacaa

$1 \longmapsto a$
$2 \longmapsto a$
$3 \longmapsto a$
$4 \longmapsto a$
$5 \longmapsto a$

$$\Sigma^k = \left\{ \text{mappings } \{1, \ldots, k\} \text{ to } \Sigma \right\}$$

Generalized Cantor's Theorem:

Injective form:

Let $h: S \to B$ be an injection, and

$$\boxed{f: S \to \text{Power}(B).}$$

Then

$$T = \{ h(s) \mid s \in S \text{ and } h(s) \notin f(s) \}$$

is not in the image of f.