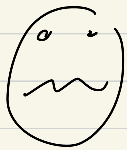
CPSC 421/501 Oct 11, 2024

- Characterization of regular languages over $\{a\}$
- Myhill-Nerode point of view of regular languages over $\{a\}$ and beyond

→ See handout:

"Non-regular languages and the Myhill-Nerode theorem"

Myhill-Nerode theorem classifies all regular languages, <u>at least in principle</u> ...



$$L = \{ a^n \mid n \geq 2,\ n \in \mathbb{N},\ \text{and } n \text{ and } n+2 \text{ are prime numbers} \}$$

Is L regular?

Easy:

L is regular iff there are

finitely many n s.t.

n, $n+2$ are prime.

iff the "twin prime

conjecture" is false

3, 5 twin primes

5, 7 " "

11, 13 " "

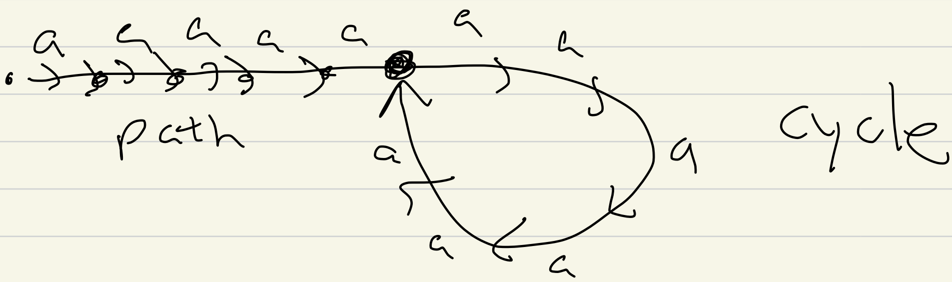
17, 19 " "

29, 31 " "

Solve this $\Rightarrow$ good position, good salary

->

Languages over $\{a\} = \Sigma$



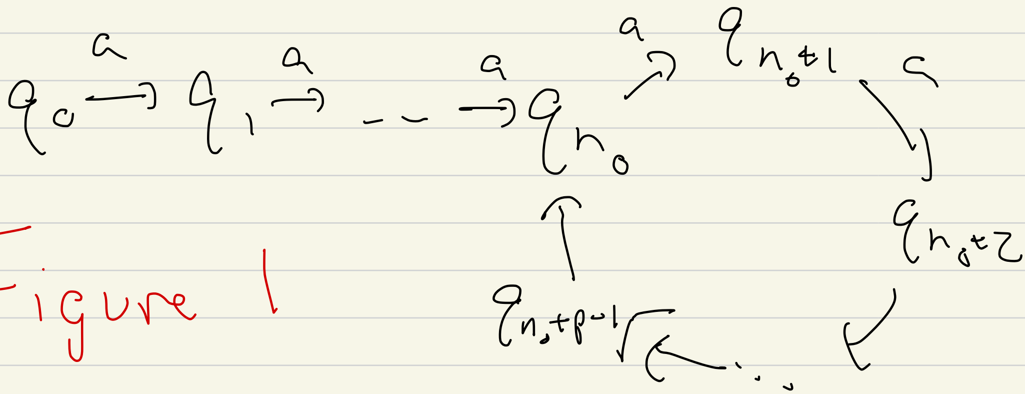


Figure 1

path length n_0 (n_0 directed edges)

cycle of p edges

$n_0 + p$ edges $\Longleftrightarrow$ $n_0 + p$ vertices
$\Longleftrightarrow$ $n_0 + p$ states

If L is regular, then there is a DFA with $n_0 + p$ states that looks like Figure 1

$\iff$

Which strings reach

q_{n_0} : $a^{n_0}, a^{n_0+p}, a^{n_0+2p}, \ldots$

if q_{n_0} is accepting:

$a^{n_0}, a^{n_0+p}, a^{n_0+2p}, \ldots \in L$

and if q_{n_0} is rejecting, not accepting, then $a^{n_0}, a^{n_0+p}, \ldots \notin L$

Similarly, for $i = 0, 1, \ldots, p-1$

q_{n_0+i} get there via

$a^{n_0+i}, \quad a^{n_0+i+p}, \quad a^{n_0+i+2p}, \ldots$

either all these $\in L$

OR " " $\notin L$

Upshot: If L is regular,

then for some n_0 and p

$n_0 \geq 0, \quad p \geq 1$

$n_0 \in \mathbb{Z}_{\geq 0}, \quad p \in \mathbb{N} = \{1, 2, \ldots\}$

for all $i = 0, \ldots, p-1$,

$$\boxed{\begin{array}{l} a^{n_0+i},\ a^{n_0+i+p},\ a^{n_0+i+2p}, \ldots \\ \text{either all these} \in L \\ \text{OR} \quad \text{"} \quad \text{"} \notin L \end{array}} \quad (*)$$

i.e.

$$\boxed{\begin{array}{c} a^n \in L \Leftrightarrow a^{n+p} \in L \\ \text{for all } n \geq n_0 \end{array}} \quad (**)$$

Result: smallest p, then smallest n_0 →

<u>Define:</u> If $L \subseteq \{a\}^*$,

then we say L is

eventually p-periodic if

$(*)$ or $(**)$ hold

for some p.

If so, the <u>smallest</u> <u>value</u>

of p for which L is

eventually is called the

<u>eventual</u> <u>period</u> of L.

e.g. if

$$\begin{tikzcd} q_0 \arrow[r] & q_1 \arrow[r] & q_2 \arrow[r, bend left] & q_3 \arrow[l, bend left] \end{tikzcd}$$

$$L = \begin{pmatrix} a^0 \\ \text{or} \\ \text{not} \end{pmatrix}, \begin{pmatrix} a^1 \\ \text{or} \\ \text{not} \end{pmatrix}, \begin{pmatrix} a^2, a^4, a^6, \ldots \\ \text{or} \\ \text{not} \end{pmatrix},$$

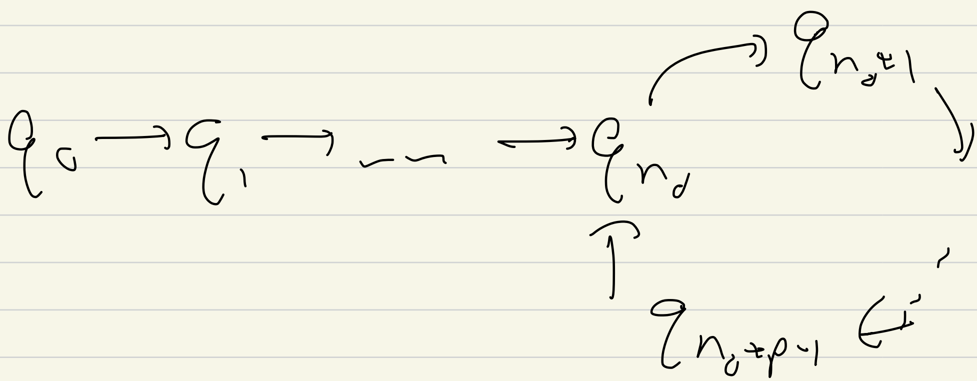
$$\text{and } \begin{pmatrix} a^3, a^5, a^7, \ldots \\ \text{or} \\ \text{not} \end{pmatrix}$$

<u>Thm</u>: If L is eventually periodic with eventual period p, and n_0 is the smallest non-neg integer

such that

$$\boxed{\forall n \geq n_0, \quad a^n \in L \iff a^{n+p} \in L}$$

The DFA recognizing L with the fewest states has n_0+p states and look like

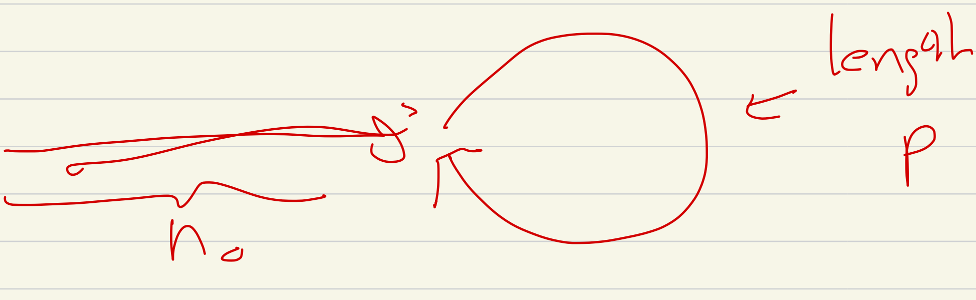


Example:

$$\{a, a^4, a^9, a^{16}, a^{25}, \ldots\}$$

$$= \{a^{n^2} \mid n \in \mathbb{N}\}$$

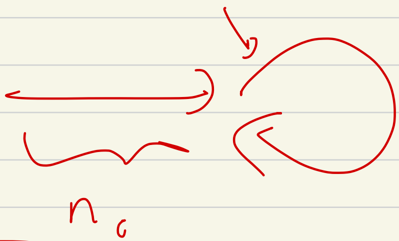
is not regular, since



$$(q+1)^2 - q^2 = 2q+1 \geq q+1$$

$$\underbrace{a^{q^2+1},\ a^{q^2+2},\ \ldots,\ a^{q^2+q}}_{\text{not in } L}$$

p



p states on the cycle

$$\boxed{\begin{gathered}\text{if } |a^{q^2+1}| \geq |a^{n_0}| \\ q^2+1 \geq n_0 \\ q \geq p\end{gathered}}$$

so all states in cycle reject

Similarly $n_1 < n_2 < n_3 < \cdots$

s.t.

$L = \{ a^{n_1}, a^{n_2}, a^{n_3}, \cdots \}$

and $(n_{i+1} - n_i) \xrightarrow{i \to \infty} \infty$

then L is not regular

Exercise (Kevin Liu)

say that

$n_1 = 3, \quad n_2 = 4, 5 \; \Big| \; n_3 = 6, \quad n_4 = 7, 8 \Big|$

$\cdots$

Could be a nice homework
or midterm problem ...

are the countably many
or uncountably "

What if $n_{i+1} = n_i + 1$
for i large

$L = \{ a^2, a^5, a^9, a^{10}, \ldots \}$

eventual period $= 1$

$n \geq 9$

$a^n \in L \iff a^{n+1} \in L$

What about if L is finite,

i.e. $\{a\}^* \setminus L$ then

this includes $a^n, a^{n+1}, a^{n+2}, \ldots$

some n large

period $= 1$

$\Rightarrow$

$\{ a^n \mid n, n+2 \text{ are primes} \}$