CPSC 421/501 Nov 15, 2024

- Reduction 1:

If 3COLOR $\in P$ gives you 10^6 USD (before taxes), then SAT $\in P$ also gives you 10^6 USD (before taxes).

Prove the Cook-Levin theorem

$\langle G \rangle$ we said

graph number of vertices #

$G = (V, E)$

$V = \{1, \ldots, n\}$

E a subset of all pairs of vertices

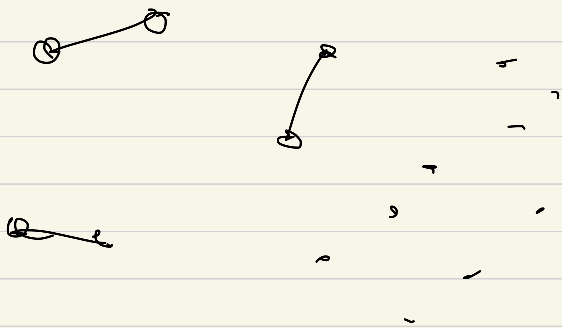
Problem: $|V| = 10^{20}$, $|E| = 3$

$\langle G \rangle = \underbrace{1000\ldots0}_{20\ 0\text{'s}} \# 3 \# 2001$

$\# 231758 \#$

$729813 \#$

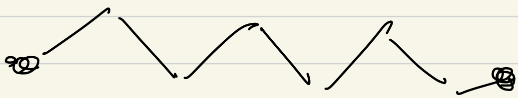
$2 \# 5$

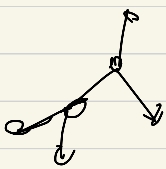


3 edges + isolated vertices

better $\rightsquigarrow$

connected graphs





tree

say

$$|V| = 100$$

and

$$|\langle G \rangle| \geq 100$$

could

(1) insist $\langle G \rangle$ gives

the adjacency matrix

$$\begin{array}{cc} & V \\ V & \left[\quad \quad \quad \right] \end{array}$$

$|V| = 100$, $|\text{Adjacency}_G|$

$\geq 100^2$

(2) Could insist: have one character for each vertex

(2') Write $|V|$ in unitary

$$8 = 8_{10}$$

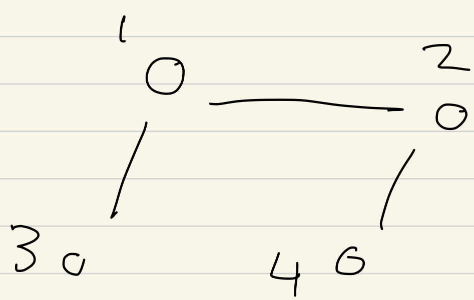
$$= 1000_2$$

$$= 11111111_1$$

Graph:

$V = \{1, 2, 3, 4\}$

$E =$

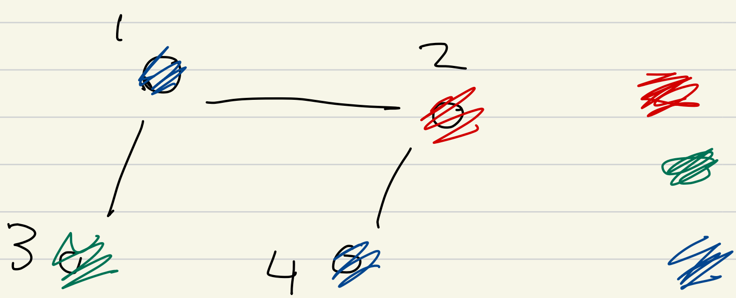


$$\langle G \rangle = \langle (V, E) \rangle$$

$$= \underbrace{1111}_{|V|=4} \# 1 \# 3 \# 1 \# 2$$

$$\# 2 \# 4$$

$$3COLOUR = \{ \langle G \rangle \mid G \text{ has a proper 3-colouring} \}$$



Claim:

$$SAT = \{ \langle f \rangle \mid f \text{ is a Boolean formula} \}$$

$$f(x_1, x_2, x_3) =$$

negation

$$((x_1 \wedge x_2) \vee \neg x_1) \wedge x_3$$

AND

OR

AND

$$\langle ((x_1 \wedge x_2) \vee \neg x_1) \wedge x_3 \rangle$$

$= ((\text{x1} \wedge \text{x2} \ldots$

or

$((\text{X sub } 1 \wedge \text{X sub } 2) \ldots$

$\in \{0, 1, \ldots, 9, \wedge, \vee, \neg, (,), X_{\text{sub}}, \}^*$

$f(x_1, \ldots, x_n)$ is <u>satisfiable</u>

if for some $x_1^* \in \{\text{True}, \text{False}\}$,

some $x_2^* \in \{\text{True}, \text{False}\}$, ...

$x_n^* \in \{\text{True}, \text{False}\}$, we have

$f(x_1^*, \ldots, x_n^*) = \text{true}$.

given

$\text{true} \wedge \text{true} = \text{true}$

$\text{true} \wedge \text{false} = \text{false}$

$\vdots$

$$SAT = \{ \langle f \rangle \mid f \text{ is a Boolean formula that is satisfiable} \}$$

$$x_1 \wedge x_2 \in SAT$$

$$x_1 \wedge (\neg x_1) \in SAT$$

$$x_1 = T, \quad T \wedge (\neg T) = F$$

$$x_1 = F, \quad F \wedge (\neg F) = F$$

Thm: If G is a graph,

$G = (V, E)$, s.t.

$V = \{1, \ldots, n\}$ for some n,

then we can build from G

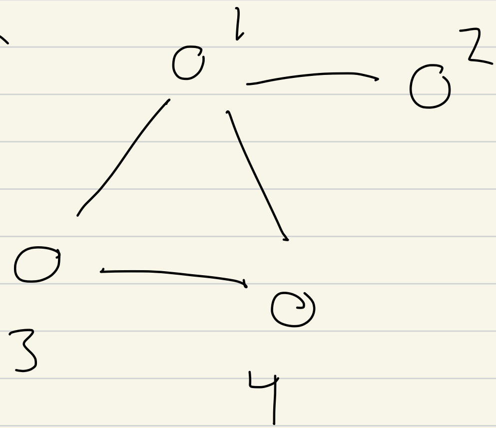
a Boolean formula, f, s.t.

1. $|\langle f \rangle| \leq \text{poly}(|\langle G \rangle|)$

and

2. $f \in SAT \iff G \in 3COLOUR$

Graph



$\langle G \rangle \in$ 3COLOUR iff

(1)

$$X_{1,red} = \left\{ \begin{array}{l} \text{does vertex 1} \\ \text{have colour red?} \end{array} \right.$$

$$X_{1,blue} = \left\{ \begin{array}{l} \text{does vertex 1} \\ \text{have colour blue?} \end{array} \right.$$

$x_{1,green} = \begin{cases} \text{does vertex 1} \\ \text{have colour green} \end{cases}$

$\vdots$

$x_{4,red} =$ etc.

$x_{4,blue} =$ etc

$x_{4,green} =$ etc.

$f = (\qquad\qquad)$ AND $(\qquad\qquad)$ AND

$(\qquad\qquad)$ AND $(\qquad\qquad)$ AND ...

$$f: \left(\begin{array}{l} X_{1,red} = T \text{ AND } X_{3,red} = T \text{ is} \\ \text{impossible} \end{array} \right)$$

AND

$$\left(X_{1,blue} \text{ -- -- } X_{3,blue} \right)$$

$$\vdots$$

$$\vdots$$

Scratch:

It can't happen that $X_{1,red} = T$

AND $X_{3,red} = T$

$$\neg (X_{1,red} \text{ AND } X_{3,red}) = \text{true}$$

$$= (\neg X_{1,red} \text{ OR } \neg X_{3,red})$$

1 ← has to have one colour among red, green, blue

$X_{1,red}$ $\quad X_{1,blue}$ $\quad X_{1,green}$:

$X_{1,red}$ OR $X_{1,blue}$ OR $X_{1,green}$ is T

iff at least one of

$X_{1,red}, X_{1,blue}, X_{1,green}$ is T

$X_{1,red}$, $X_{1,blue}$ can't both be true