CPSC 421/501 Nov 20, 2024

- NP non-deterministic poly time
- SAT, 3SAT, 3COLOUR $\in$ NP
- Define $A \le_{\substack{\text{poly}\\ \text{time}}} B$ formally.

Examples: 3COLOUR $\le_p$ SAT,

3COLOUR $\le_p$ 3SAT,

3SAT $\le_p$ 3COLOUR (homework)

- Cook-Levin Theorem: If $L \in$ NP, then $L \le_{\text{poly}}$ 3SAT

DFA $(Q, \Sigma, \delta, q_0, F)$

$$\delta : Q \times \Sigma \to Q$$

NFA $\delta : Q \times \Sigma \to \text{Power}(Q)$

Turing machine $(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$$

Non-deterministic T.M. :

$$\delta : Q \times \Gamma \to \text{Power}(Q \times \Gamma \times \{L, R\})$$

Deterministic	Non-Deterministic

Initial Config

$\downarrow$

2nd Config

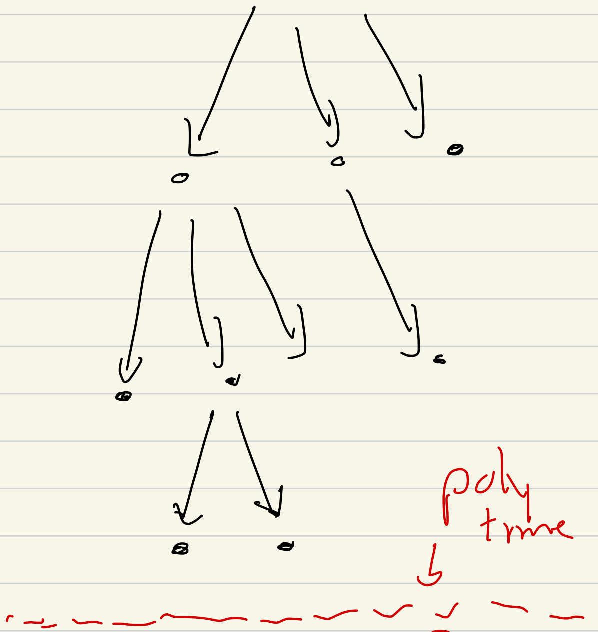
$\downarrow$

$\vdots$

$\downarrow$

Last Config

Initial



A non-deterministic machine <u>accepts</u> an input if there is <u>at least one</u> <u>accepting</u> path

non-det
poly
time

$$NP = \left\{ L \;\middle|\; \begin{array}{l} \text{there is a non-det T.m., } M \text{, s.t.} \\ \text{① all computation paths halt in time } \leq Cn^k \text{ on all inputs of length } n \\ \text{② } \forall w \in \Sigma^* \\ \quad w \in L \iff M \text{ accepts } w \end{array} \right\}$$

e.g. SAT, 3SAT, 3COLOUR $\in NP$

Remark: In Cn^k time steps,

there could be

$$|Q \times \Gamma \times \{L, R\}|^{Cn^k}$$

possible paths ...

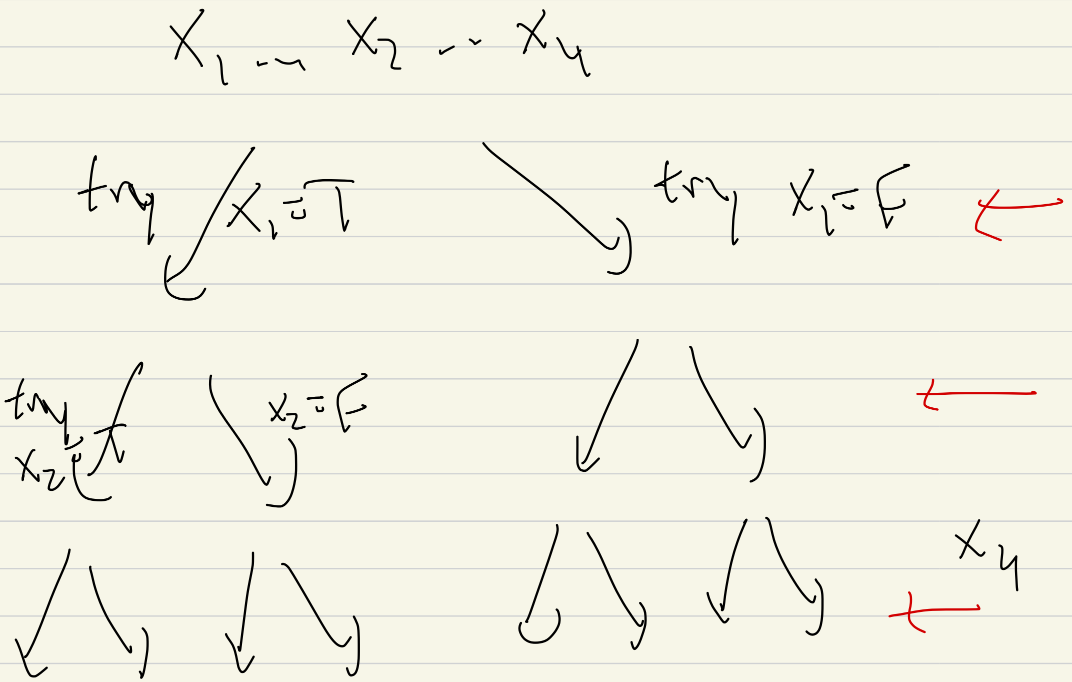
Remark: To see that $SAT \in NP$

$$\varphi = (x_1 \vee x_2) \wedge \neg(\neg x_1 \vee x_4)$$

we can "try" all $8 = 2^3$ settings

of $x_1, x_2, x_4 \to \{F, T\}$

Explain SAT ∈ NP, 3 COLOUR ∈ NP



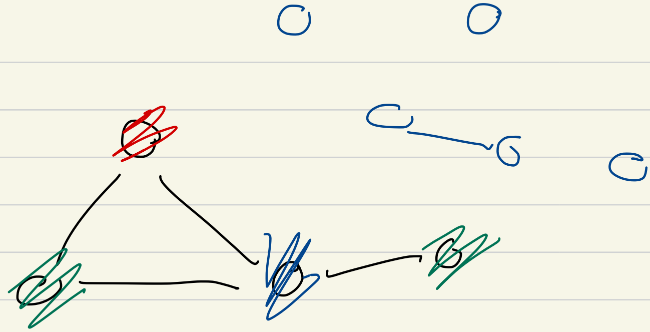
. . . . extra computation

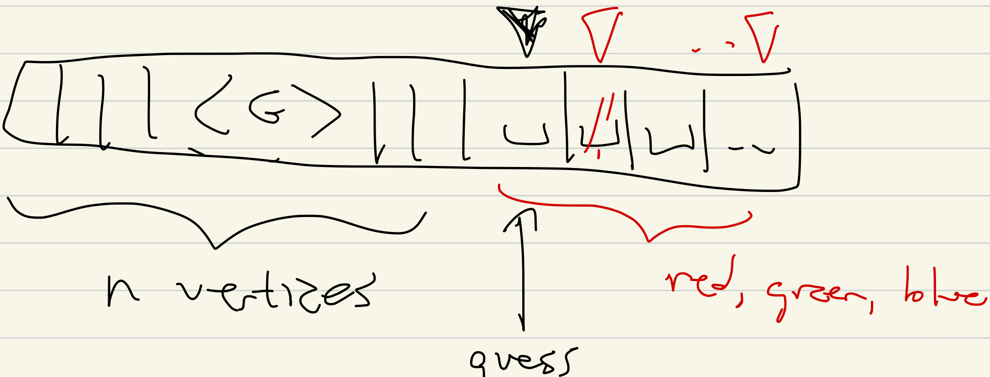
$2^3 = 8$ "results"

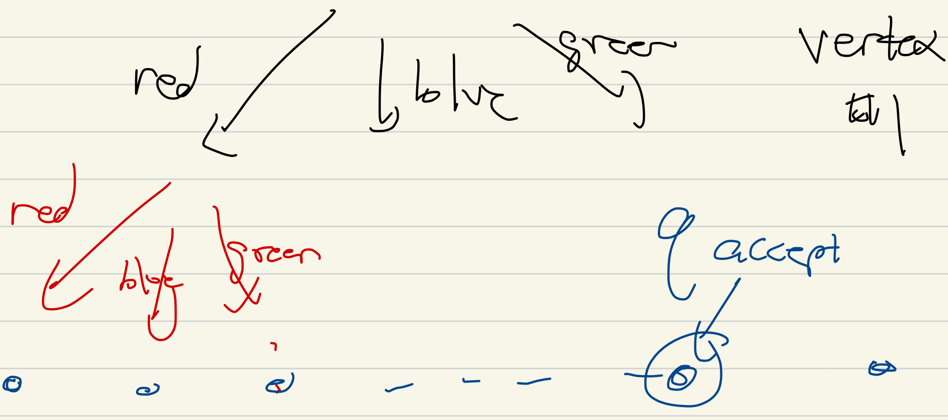
If one of .... results is accept, then we accept φ

3COLOUR:

description







$$\Sigma, \quad \Gamma = \Sigma \cup \{\text{red}, \text{green}, \text{blue}\} \ldots$$

Formal Definition: If $A \subset \Sigma_1^*$, $B \subset \Sigma_2^*$, we write

$$A \leq_p B \quad \text{or} \quad A \leq_{\substack{\text{poly}\\\text{time}}} B$$

if there is $f : \Sigma_1^* \to \Sigma_2^*$ s.t.

1. f can be computed in poly time,
2. $\forall w \in \Sigma_1^*$

$$w \in A \iff f(w) \in B$$

E.g. $3COLOUR \leq_p SAT$,

$3COLOUR \leq 3SAT$, $\quad 3SAT \leq 3COLOUR$

$$3COLOUR \leq_p 3SAT$$

graph

poly time

$$\Sigma_1^* \xrightarrow{f} \Sigma_2^*$$

$$\langle G \rangle \in 3COLOUR$$

iff

$$\langle f(G) \rangle \in 3SAT.$$

Definition: B is NP-complete if ① $B \in NP$,

② $\forall A \in NP$, $A \leq_{\substack{\text{poly}\\\text{time}}} B$

Theorem: [Cook-Levin] If $L \in NP$, then $L \leq_p SAT$;

therefore SAT is NP-complete.

Note: Our proof shows that $L \leq_p 3SAT$.

Lemma: Fix $a_1, a_2, a_3, a_4 \in \{T, F\}$

$$a_1 \text{ or } a_2 \text{ or } a_3 \text{ or } a_4 = T$$

iff

$$\left.\begin{array}{l}(a_1 \text{ or } a_2 \text{ or } z) \text{ AND} \\ (\neg z \text{ or } a_3 \text{ or } a_4)\end{array}\right\} \text{Homework}$$

is satisfiable

Midterm median 81.08%

Corollary: 3COLOUR is NP-complete.

we didn't mention this on Nov 20