CPSC 421/501 Nov 22, 2024

- The Cook-Levin theorem:

If $L \in NP$, then $L \le_p 3SAT$.

- Next week!

Chapter 8: Space complexity,

$\underline{P}\,S\underline{PACE} = \underline{NP}\ \underline{SPACE}$

(is true, and not worth \$$10^6$ USD before taxes)

- Lemma 1: If $f: \{F, T\}^d \to \{F, T\}$, then f can be written in d-CNF form, with at most 2^d clauses.

- Remark: For any a_1, a_2, a_3, a_4 in $\{F, T\}$

$$a_1 \text{ OR } a_2 \text{ OR } a_3 \text{ OR } a_4 \cancel{= T}$$

$$\iff$$

$$(a_1 \text{ OR } a_2 \text{ OR } z_1) \text{ AND}$$

$$(\neg z_1 \text{ OR } a_3 \text{ OR } a_4)$$

is satisfiable.

Proof:

What if $a_1 = T$: why is

$$\overbrace{(\overset{T}{a_1} \text{ OR } \overset{?}{a_2} \text{ OR } \overset{?}{z_1})}^{\to T}$$

AND

$$\overbrace{(\overset{T}{\neg z_1} \text{ OR } \overset{?}{a_3} \text{ OR } \overset{?}{a_4})}^{\to T}$$

T

satisfiable?

$a_1 = T, \quad a_2, \ldots, a_4 = ?$

$z_1 = F$

- Corollary: Any φ in 4-CNF form has a $\hat{\varphi}$ in 3-CNF form s.t.

① $\varphi \in SAT \Leftrightarrow \tilde{\varphi} \in SAT$

② $(\#\text{ clauses in } \hat{\varphi})$

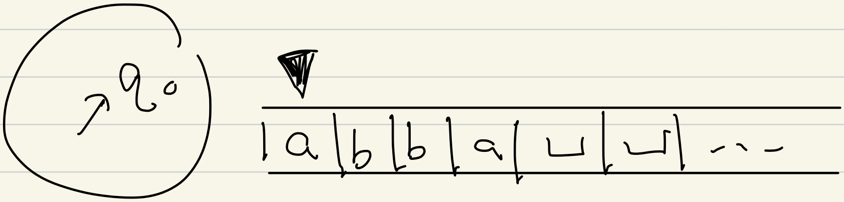
$$\leq 2(\#\text{ clauses in } \varphi)$$

- Homework 9: Generalize this...

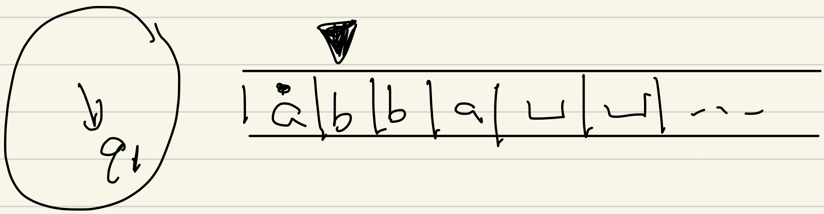
Proof of Cook-Levin: Say M is non-det Turing m. running in time n^k

input here: $w = abba$ (here)

initial

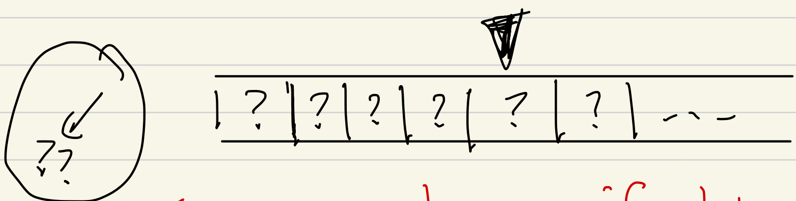


step 2



⋮

step n^k



← we only care if state q_{acc}

Rem: ▼ at time n^k at cell $\leq n^k$

$$X_{ij\gamma} = \begin{cases} T & \text{if at step } i\text{, cell } j \text{ contains} \\ & \text{the symbol } \gamma\text{, and} \\ F & \text{otherwise} \end{cases}$$

$$Y_{ij} = \begin{cases} T & \text{if at step } i\text{, tape head is} \\ & \text{over cell } j \\ F & \text{otherwise} \end{cases}$$

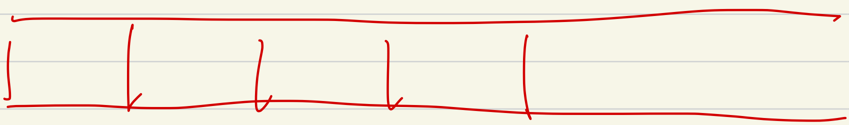
$$Z_{iq} = \begin{cases} T & \text{if at step } i\text{, we are in} \\ & \text{state } q\text{,} \\ F & \text{otherwise} \end{cases}$$

$$\overbrace{i, j \in \{1, \dots, n^k\}}^{n^k \text{ possible } i,j}, \quad \overbrace{\gamma \in \Gamma}^{|\Gamma| \text{ possible } \gamma}, \quad \overset{|Q| \text{ possible } q}{q \in Q}$$

Say M is fixed, k is fixed

w = anything

= say, abba



$\Gamma = \{a, b, \sqcup, \dot{a}, \dot{b}, 4\}$

$i=1, j=1$

$X_{ij\gamma}$

time 1, cell 1:

- we see a
- ″ ″ b
- ″ ″ $\sqcup$
- ″ ″ $\dot{a}$
- ″

$X_{11a}, X_{11\sqcup}, X_{11b}, X_{11\dot{a}}, X_{11\dot{b}}, X_{114}$

Encode non-determinism

(step 1 have to see correct initial state)

AND

(step 1 to step 2 transition has to be valid)

$\varphi_{1 \to 2}$

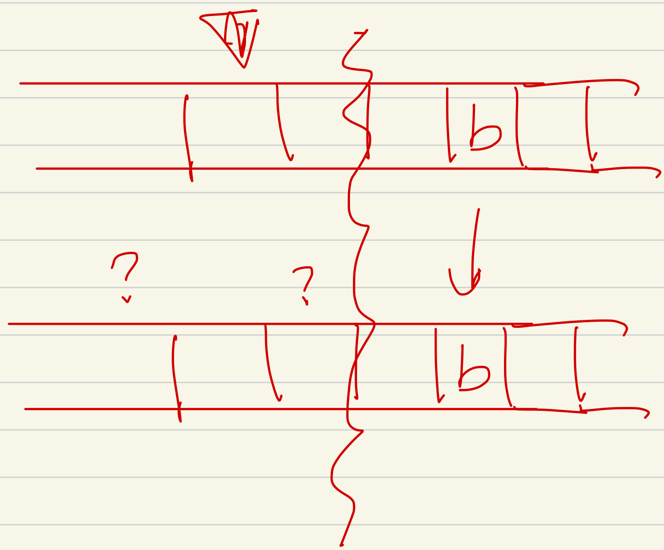
AND

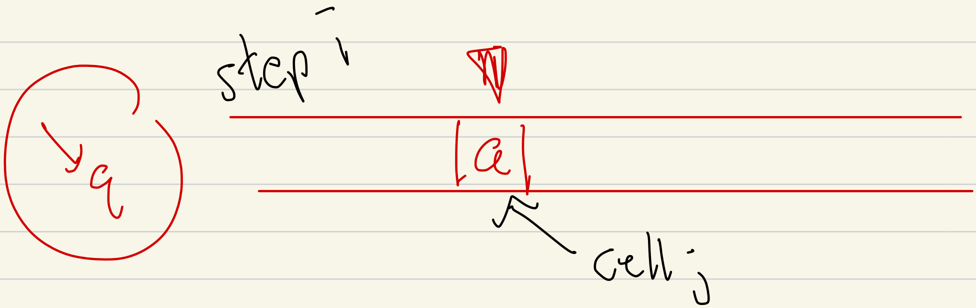
(2 3)

⋮

$\varphi_{k \to k+1}$

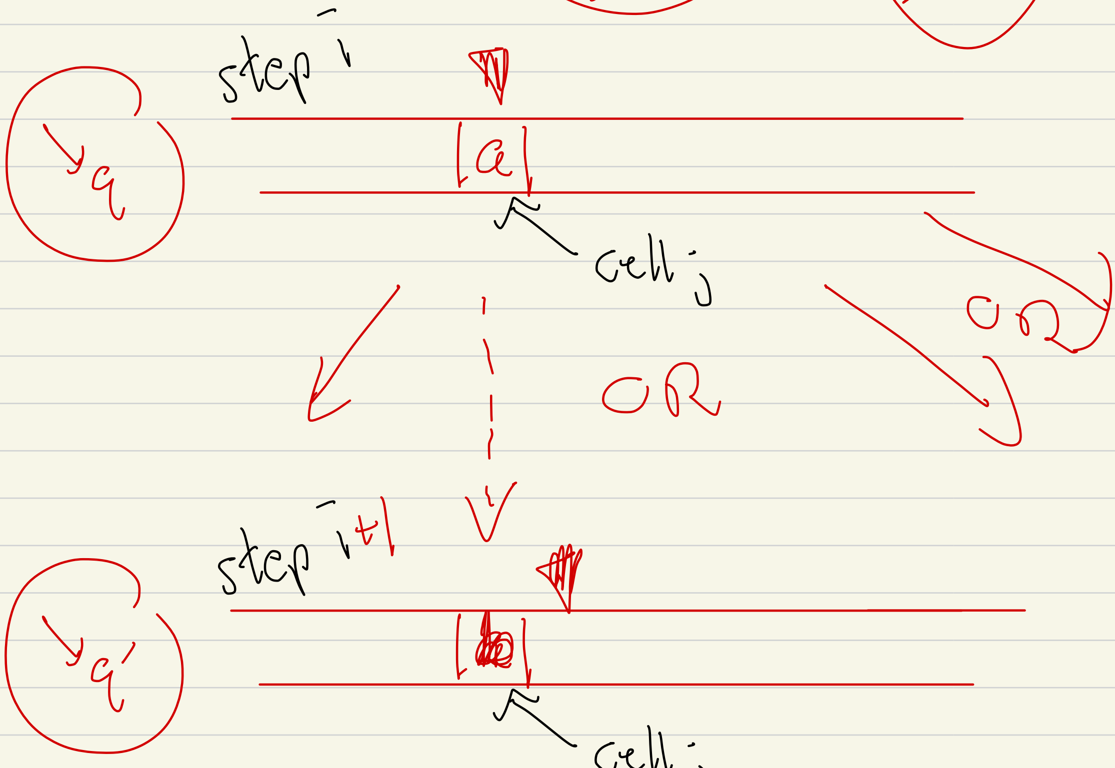
tape head is far





$$\delta(q, a) \to \begin{cases} (q', b, R) \\ (q'', b, L) \\ \ldots \end{cases}$$

$x_{i,j,a} = T$, $\cancel{x_{i,j,b} = F, \ldots}$ $\quad$ $y_{ij} = T$, $\cancel{x_{i,j',*} = F}$ $\quad$ $z_{iq} = T$



$x_{i+1,j,a} = ?$ $\quad$ $x_{i+1,j,b} = ?$, ...

$$\left(X_{i,j,a} \text{ and } Y_{ij} \text{ and } Z_{iq} \right) \Rightarrow$$

$$\left(\begin{array}{l} X_{i+1,j,b} = \text{true} \quad \text{AND} \\ Y_{i+1,j+1} = \text{true} \quad \text{AND} \\ Z_{i+1,q'} = \text{true} \end{array} \right) \text{ OR}$$

$$\left(\qquad \qquad \right) \text{ OR}$$

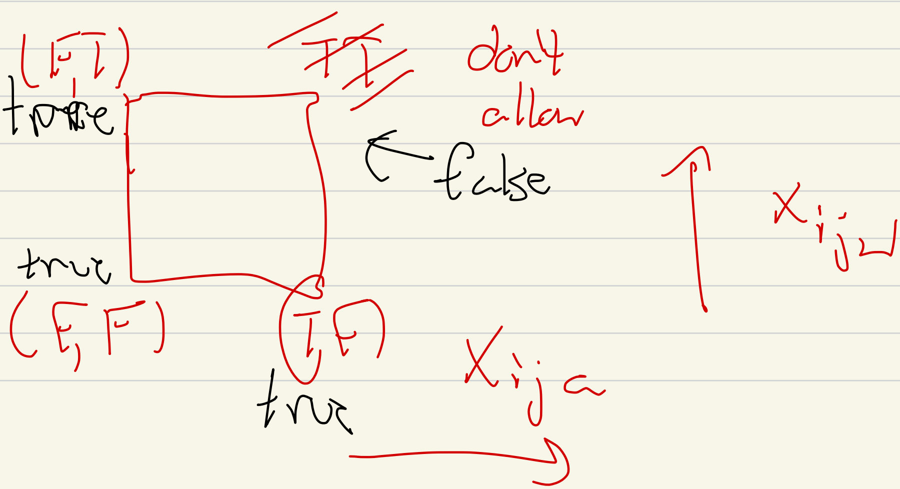
$\vdots$

$\forall i, j$

$X_{ija} = \text{True} \Rightarrow X_{ij\sqcup} = F$

$i \in [n^k], \; j \in [n^k] \quad a, \sqcup \in \Gamma$

$X_{ija}, \; X_{ij\sqcup} \quad$ condition



2-CNF

$(\neg x_{ija} \;\text{OR}\; \neg x_{ij\sqcup})$

(OR) AND

(OR) AND

(OR) AND

(OR)