# (DRAFT:) TDA, POINT CLOUDS, AND SOME POINT-SET TOPOLOGY

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**Disclaimer:** The material may sketchy and/or contain errors, which I will elaborate upon and/or correct in class. For those not in CPSC 531F: use this material at your own risk...

**Notes:** In class I will often give (extra) examples and draw pictures to clarify and provide intuition for the ideas in this article.

We often use *italics* for precise mathematical terms that we have not (yet) defined; we use "quotation marks" to delimit terms that are vague or whose precise definitions depends on the context and/or author(s). [A sentence or phrase in square brackets is not essential to the rest of the article.]

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# 1. Introduction to TDA (Topological Data Analysis) and Motivation via Point Clouds

This section motivates everything that will follow in the course for roughly the first month or two. However, it is not a course prerequisite that you have seen all the mathematical terms in this section. For this reason, this section is independent of the rest of this article, and we will draw some pictures in class (and "wave our hands" a lot) when we discuss this section. [By the end of this course you should understand everything in this section.] The definitions and details start in Section 2.

The point of this article is to introduce TDA as motivated by "point clouds," and then "review" the ideas from point-set topology that we will need. We begin the course by studying "point clouds;" often it suffices to define a *point cloud* as a finite subset $\mathcal{P}\subset\mathbb{R}^n$. Here we think of $n$ as fixed, although we may have numerous point clouds $\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_m\subset\mathbb{R}^n$.

## 1.1. Point Clouds as Models.

Assume that $\mathcal{P}\subset\mathbb{R}^n$ is a finite set (or "point cloud") that represents a "sample" of points from a subset $X\subset\mathbb{R}^n$ (possibly a "noisy sample," meaning $\mathcal{P}$ is not necessarily a subset of $X$). Then one can try to use $\mathcal{P}$ to infer the *topological invariants* of $X$ (e.g., its homotopy groups, (co)homology groups, compactly supported cohomology groups, etc.). This is one example of what one calls *topological data analysis* (TDA).

For the first month or so of this course we study the following method:

(1) For each real $\delta>0$, we define what it means for a subset of points $A\subset\mathbb{R}^n$ to be "$\delta$-close (to one another)." One popular choice is that $A$ must be contained in a ball (likely a closed ball[1]) n $\mathbb{R}^n$ of radius $\delta$; another is that the diameter of $A$ is at most $\delta$ (i.e., any two points of $A$ are within a distance $\delta$).

(2) For each real $\delta>0$, one defines $\mathcal{K}_{\rm small}(\mathcal{P},\delta)\subset\mathbb{R}^n$ to be the union of the convex hulls[2] of subsets $A\subset\mathcal{P}$ that are $\delta$-close.

(3) One fixes a map $f\colon\mathcal{P}\to\mathbb{R}^N$ such that $f(\mathcal{P})$ are in "general position" (i.e., the vectors $f(P_1)-f(P_2)$ for $P_1,P_2$ ranging over $\mathcal{P}$ span a subspace of $\mathbb{R}^N$ of dimension $|\mathcal{P}|-1$). (Hence $N\ge|\mathcal{P}|-1$.) For each real $\delta>0$, one defines $\mathcal{K}_{\rm big}(\mathcal{P},\delta)\subset\mathbb{R}^N$ to be the union of the convex hulls of subsets $f(A)\subset\mathbb{R}^N$ of the subset $A\subset\mathcal{P}$ that are $\delta$-close.

(4) Often $\mathcal{P}\subset\mathbb{R}^n$ is fixed, and hence we will simply write $\mathcal{K}_{\rm small}(\delta)$ and $\mathcal{K}_{\rm big}(\delta)$. When we compute *simplicial homology* and related invariants we will focus on $\mathcal{K}_{\rm big}(\delta)$ alone (at times we may be hoping that $\mathcal{K}_{\rm big}(\delta)$ is *homotopy equivalent* to $\mathcal{K}_{\rm small}(\delta)$).

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[1] In $\mathbb{R}^n$ or a metric space there is a notion of an *open ball* and of *closed ball*. One often writes just "ball," since it is usually clear which one means; for example, when working with open sets, the term "ball" is likely to mean an open ball;" by contrast, if you are finding the Laplacian eigenvalues of an open, bounded subset of $\mathbb{R}^n$, you are likely to consider a minimizing sequence for the Dirichlet integral; you then need the fact that "the unit ball of a separable Hilbert space is weakly compact;" here you are talking about the closed unit ball. If you are using a floating point calculations on a computer, you likely can't tell the difference between an open ball in $\mathbb{R}^n$ and the corresponding closed ball...

[2] Recall that the *convex hull* of a set $S\subset\mathbb{R}^n$ is the intersection of all convex subsets of $\mathbb{R}^n$ containing $S$; hence if $A=\{\mathbf{a}_1,\ldots,\mathbf{a}_m\}$ if finite, then the convex hull of $A$ is the set of convex linear combinations of the elements of $A$, i.e., all vectors of the form $\alpha_1\mathbf{a}_1+\cdots+\alpha_m\mathbf{a}_m$ where the $\alpha_i$ are non-negative real numbers (i.e., $\alpha_i\in\mathbb{R}_{\ge0}$) whose sum is 1 (i.e., $\alpha_1+\cdots+\alpha_m=1$).

**Remark 1.1.** The set $\mathcal{K}_{\mathrm{big}}(\delta)$ in (3) depends on the choice of $N$ and the map $f\colon \mathcal{P} \to \mathbb{R}^N$. However, the resulting *topological space*, $\mathcal{K}_{\mathrm{big}}(\delta)$, doesn't depend on $f$. If we are considering more than one point cloud, e.g., we are given $\mathcal{P}_1, \ldots, \mathcal{P}_m \subset \mathbb{R}^n$, then it can be convenient to have all the $\mathcal{K}_{\mathrm{big}}(\mathcal{P}_i, \delta)$ lie in the same space $\mathbb{R}^N$ rather than each $\mathcal{K}_{\mathrm{big}}(\mathcal{P}_i, \delta)$ lying in its own $\mathbb{R}^{N_i}$. This is one reason to leave $f$ as a general map with $f(\mathcal{P})$ in general position.

**Remark 1.2.** More generally, we can let the property of $\delta$-closeness depend on $\delta$ *and* $\mathcal{P}$. Moreover, if we think that $\mathcal{P}$ models a space $X \subset \mathbb{R}^n$ be we are unsure what $X$ is, then we could let $\delta$-closeness depend on $X$, i.e., $\mathcal{K}_{\mathrm{small}}(\mathcal{P}, \delta) = \mathcal{K}_{\mathrm{small}}(X, \mathcal{P}, \delta)$ and $\mathcal{K}_{\mathrm{big}}(\mathcal{P}, \delta) = \mathcal{K}_{\mathrm{big}}(X, \mathcal{P}, \delta)$.

**Remark 1.3.** To define *persistent homology*, will need the condition that if $A \subset \mathbb{R}^n$ is "$\delta$-close," then it is also "$\delta'$-close for $\delta' > \delta$. If so, then $\mathcal{K}_{\mathrm{small}}(\delta) \subset \mathcal{K}_{\mathrm{small}}(\delta')$ and $\mathcal{K}_{\mathrm{big}}(\delta) \subset \mathcal{K}_{\mathrm{big}}(\delta')$.

**Remark 1.4.** Note that if $A \subset \mathbb{R}^n$ lies in some ball of radius $\delta$, then so does $A'$ whenever $A' \subset A$. Similarly if the diameter of $A$ is at most $\delta$. This property is computationally convenient for our purposes: namely, whether or not the convex hull of $A'$ lies in $\mathcal{K}_{\mathrm{small}}(\delta)$, or that of $f(A')$ lies in $\mathcal{K}_{\mathrm{big}}(\delta)$, is instrinsic to $A'$ and doesn't depend on whether or not some $A \subset \mathcal{P}$ containing $A'$ is considered to be $\delta$-close.

In this way we have a collection $\{\mathcal{K}_{\mathrm{small}}(\delta)\}_{\delta \in \mathbb{R}_{>0}}$ of subsets of $\mathbb{R}^n$, and another collection $\{\mathcal{K}_{\mathrm{big}}(\delta)\}_{\delta \in \mathbb{R}_{>0}}$ of subsets of $\mathbb{R}^N$ and such that:

(1) for $\delta > 0$ small, $\mathcal{K}_{\mathrm{small}}(\delta)$ and $\mathcal{K}_{\mathrm{big}}(\delta)$ consist of $|\mathcal{P}|$ distinct points, which is not per se (in of itselft) interesting;
(2) for $\delta > 0$ large, $\mathcal{K}_{\mathrm{small}}(\delta)$ and $\mathcal{K}_{\mathrm{big}}(\delta)$ are, respectively, the convex hulls of $\mathcal{P}$ and of $f(\mathcal{P})$, which is again not per se interesting;
(3) however, hopefully for $\delta$ of "moderate size," $\mathcal{K}_{\mathrm{small}}(\delta)$, $\mathcal{K}_{\mathrm{big}}(\delta)$, and $X$ all have the same values for a number of interesting "topological" invariants.

**Remark 1.5.** Before doing any topology we encounter a serious problem: $X$, $\mathcal{P}$, and $\mathcal{K}_{\mathrm{small}}(\delta)$ for $\delta > 0$ are the subsets of $\mathbb{R}^n$ that are of most interest to us. However, the topological invariants $\mathcal{K}_{\mathrm{big}}(\delta)$ are easiest to compute, because they can be reduced to the combinatorics of knowing which subsets $A \subset \mathcal{P}$ are $\delta$-close. The fact that $f(\mathcal{P}) \subset \mathbb{R}^N$ are in general position gives a simple (affine linear) map $\mathbb{R}^N \to \mathbb{R}^n$, which restricts to a map $\mathcal{K}_{\mathrm{big}}(\delta) \to \mathcal{K}_{\mathrm{small}}(\delta)$. However, there is no a priori guarantee that this map can be paired with a map $\mathcal{K}_{\mathrm{small}}(\delta) \to \mathcal{K}_{\mathrm{big}}(\delta)$ that together form a homotopy equivalence.

1.2. **Persistent Homology and Barcodes.** The idea of *persistent homology* is that it is more practical and effective to replace (3) above by: (3′) (again, hopefully...) there is a small $\delta_1$ and large $\delta_2$ such that the "topological invariants" of interest to us (e.g., homotopy groups, homology groups) of $X$ are, roughly speaking, those of $\mathcal{K}_{\mathrm{big}}(\delta_1)$ that "persist" in $\mathcal{K}_{\mathrm{big}}(\delta_2)$. To understand what this means, recall that homotopy groups and homology groups are invariants — actually groups — associated any topological space such that if $X \to Y$ is a continuous map, then the value of each homotopy group or homology group of $X$ maps (as a group) to that of $Y$. Since $\mathcal{K}_{\mathrm{big}}(\delta_1) \subset \mathcal{K}_{\mathrm{big}}(\delta_2)$ in the definitions above for $\delta_1 < \delta_2$, this gives a sense to "persistent homology," i.e., a maximal subgroup of $H_i(\mathcal{K}_{\mathrm{big}}(\delta_1))$ (not uniquely determined) that maps bijectively onto the image of $H_i(\mathcal{K}_{\mathrm{big}}(\delta_1))$ in $H_i(\mathcal{K}_{\mathrm{big}}(\delta_2))$.

More generally, there is a "barcode theorem" that implies that any sequence of vector spaces, with linear maps from each vector space to the next one in sequence, decomposes into direct sums of pieces, each of which "is created" or "begins to live" somewhere in the sequence, and persists until it is mapped to 0 somewhere "later" in the sequence. In this course we will formalize the "barcode theorem;" this is quite an interesting theorem in its own right. Since the $\mathcal{K}_{\mathrm{big}}(\delta)$ takes on only finitely many values for all $\delta \in \mathbb{R}_{>0}$, the $i$-th homology group, $H_i(\mathcal{K}_{\mathrm{big}}(\delta))$ can be viewed as a finite sequence of groups with a map from each group to the next one. If we compute the $H_i$ with coefficients in a field $\mathbb{F}$, then the $H_i(\mathcal{K}_{\mathrm{big}}(\delta))$ form a sequence of $\mathbb{F}$-vector spaces.

In any event, this allows us to replace (3) and (3′) above with: (3″) for each $i \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$, as $\delta$ varies over $\mathbb{R}_{>0}$, we compute the "bars" in $H_i(\mathcal{K}_{\mathrm{big}}(\delta))$, i.e., the $i$-th homology group of $\mathcal{K}_{\mathrm{big}}(\delta)$, and look for "bars" that persist over a wide range of values of $\delta$.

1.3. **Learning Algebraic Topology.** Most books in algebraic topology begin by defining the *fundamental group*, $\pi_1(X)$, of a *topological space*, $X$, since it is simple to define and has many applications. You do need to define what is meant by a *homotopy* between two paths in a topological space; however, anyone working with homology groups will eventually want to understand that homology groups are the same for any two topological spaces that are homotopy equivalent. Hence the notion of homotopy is fundamental to homology groups (as well as to the homotopy groups, $\pi_i(X)$ of a space $X$).

From there, some textbooks will discuss higher homotopy groups, i.e., $\pi_i(X)$ for $i \geq 2$ ($\pi_0(X)$ just measures the number of connected components of $X$), and discuss (or complain about...) how difficult these are to compute. Whether or not one discusses higher homotopy groups, most textbooks then discuss the *(singular) homology groups*, $H_i(X)$, of a topological space. For example, $H_0(X)$ measures the number of connected components of $X$, and $H_1(X)$ turns out to be the abelianization of $\pi_1(X)$; however, homology groups are very different: for example, $H_i(X) = 0$ whenever $i$ is larger than the "dimension" of $X$, and the homology groups $H_i(X)$ are typically much easier to compute than homotopy groups $\pi_i(X)$ (for general or "large" $i$).

[Moreover, if a topological space $X$ is endowed with the structure of a smooth $n$-dimensional manifold, then one can define differential forms on $X$, and generally Riemannian metrics on $X$; this field is generally called "differential geometry," and is a vast field with many examples to provide intuition, and famous applications in physics. In particular, *de Rham coholomogy* is a (co)homology theory based on differential forms which is easy to define and work with; the dual groups are homology groups that agree with (the torsion-free part of) the singular homology groups.]

If you are mainly interested in the homology groups, $H_i(X)$, of a topological space, and don't want to bother with homotopy groups, then there are two options: (1) some excellent textbooks on algebraic topology first discuss homology groups from the get go, like Massey's textbook [Mas80]; (2) many textbooks, such as Hatcher's [Hat02] are quite readible if you skip the earlier sections on homotopy theory and start reading from the definitions of simplicial and singular homology groups (e.g., Section 2.1 of Hatcher's textbook).

A less common option is that some books begin directly by describing the *simplicial homology groups* of a *simplicial complex* or, more generally, a $\Delta$-*complex* (in the sense of Hatcher [Hat02], Section 2.1). One then develops some intuition for what homology groups are measuring, and one takes things from there. A terrific example of this is [Mun84], that has many pictures and examples, or — free for UBC students — Armstrong's textbook [Arm83] (see Chapter 8, although I'm not sure how much Chapter 8 is independent of previous material).

Many papers in TDA currently seem to define *simplicial homology groups* of *simplicial complexes* and seemingly imply that you are supposed to understand what is going on from these bare-bones descriptions. This is much like reading a research paper in some field that defines a *derivative* or a *linear transformation*. Of course, there is nothing wrong with setting up your particular notation, whether it's for calculus, linear algebra, or algebraic topology. However, any research paper making active use of calculus or linear algebra will likely seem magical and/or mysterious (if not completely inaccessible) unless you've already seen the basics in these fields. Similarly for algebraic topology.

1.4. **The Setting of Algebraic Topology.** The simplest way to understand (co)homology is through differential forms in $\mathbb{R}^n$, which gives rise to differential form on smooth (or sufficiently differentiable) manifolds. It's not clear we'll have much time to discuss this, although most intro courses in differential geometry will devote some time to this topic and provide valuable intuition.

The usual setting of basic Algebraic Topology is *topological spaces* and *continuous maps*. This is a simpler setting than that of, say, smooth manifolds; the tradeoff is that it is more difficult to define (co)homology there.

Part of what we do in algebraic topology will be done for subsets of $\mathbb{R}^n$ (e.g., spaces like $\mathcal{K}_{\rm small}(\delta)$ and $\mathcal{K}_{\rm big}(\delta)$), and in this case it suffices to understand the topology of $\mathbb{R}^n$ (which we cover in the next section).

However, ultimately we will work we work with topological spaces for a number of reasons, such as: (1) some of our topological spaces do not arise as subsets of $\mathbb{R}^n$, and (2) even when they do, i.e., we work with a topological space that is homeomorphic (i.e., isomorphic as a topological space) to a subset $X \subset \mathbb{R}^n$ (in its induced topology), it can be tedious to keep track of an $X \subset \mathbb{R}^n$ isomorphic to the topological space we have in mind when we perform certain operations in algebraic topology.

Since subsets of $\mathbb{R}^n$ provide a lot of good intuition about topological spaces and continuous maps, the next section discusses basic topology (often called "point-set" topology) in $\mathbb{R}^n$.

## 2. Topology in $\mathbb{R}^n$

The notation $\mathbb{R}^n$ connotes that $n \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$. Some of the terminology in this section will not be defined or explained here; at UBC you will have seen these terms in Math 320.

2.1. **Functions on Compact Subsets.** One of the most important facts of advanced calculus (a.k.a., real variables) (see, for example, [Fri71]) is that any continuous function on a closed and bounded subset of $\mathbb{R}^n$ attains a maximum value (and a minimum value) somewhere. This is Theorem 2.2. Understanding this theorem and related notions will motivate everything else we do.

We use the notation and definitions.

(1) We use the notation $\mathbf{x} = (x_1, \ldots, x_n)$ for elements of $\mathbb{R}^n$, and the Euclidean norm (a.k.a. the $L^2$ or $\ell^2$ norm) on $\mathbb{R}^n$:
$$|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}.$$
(2) If $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{x}_1, \mathbf{x}_2, \ldots$ is a sequence of points in $\mathbb{R}^n$, we say that $\{x_i\}_{i\in\mathbb{N}}$ (or just $x_i$) *converges to* $\mathbf{x}$ if for any $\epsilon > 0$ there is an $i_0$ for which $|\mathbf{x}_i - \mathbf{x}| < \epsilon$ for $i \geq i_0$.
(3) A subset $X \subset \mathbb{R}^n$ is *closed* if for every sequence $\mathbf{y}_1, \mathbf{y}_2, \ldots$ in $X$ that converges to some $\mathbf{y} \in \mathbb{R}^n$ we have that $\mathbf{y} \in X$.
(4) Let $X \subset \mathbb{R}^n$. A function $f \colon X \to \mathbb{R}^m$ is *continuous* if either of the following equivalent conditions hold:
   (a) for every sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ in $X$ converging to an $\mathbf{x} \in X$ we have that $f(\mathbf{x}_i) \to f(\mathbf{x})$;
   (b) for every $\mathbf{x}_0 \in X$ and real $\epsilon > 0$ there is a $\delta > 0$ such that if $\mathbf{x} \in X$ and $|\mathbf{x} - \mathbf{x}_0| < \delta$ we have $|f(\mathbf{x}) - f(\mathbf{x}_0)| < \epsilon$.

   (EXERCISE: If you don't know that these are equivalent, then prove this.)
(5) A subset $X \subset \mathbb{R}^n$ is *bounded* if for some real $M$ we have $\mathbf{x} \in X$ implies that $|\mathbf{x}| \leq M$.

**Example 2.1.** Let $f \colon \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. Then $f^{-1}(\mathbf{0}) = \{\mathbf{x} \mid f(\mathbf{x}) = \mathbf{0}\}$ is closed. Since the function $\mathbf{x} \mapsto |\mathbf{x}|$ is continuous, it follows that for any $n \in \mathbb{N} = \{1, 2, \ldots\}$,
$$\mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| = 1\}$$
is closed. Clearly $\mathbb{S}^{n-1}$ is also bounded.

The following theorems is fundamental to a vast amount of mathematics.

**Theorem 2.2.** *Let $X \subset \mathbb{R}^n$ be a closed and bounded subset of $\mathbb{R}^n$, and let $f \colon X \to \mathbb{R}$ be continuous. Then $f$ has a maximum value, i.e., there exists an $x^* \in X$ such that $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all $\mathbf{x} \in X$. (Similarly with minimum replacing maximum.)*

Let us outline the proof.

**Definition 2.3.** A subset, $X \subset \mathbb{R}^n$, is *compact* if every sequence in $X$ has a convergent subsequence.

**Theorem 2.4** (Heine-Borel)**.** *A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.*

Assuming the Heine-Borel theorem, to prove Theorem 2.2 we, roughly speaking, take a "maximizing sequence," $\mathbf{x}_1, \mathbf{x}_2, \ldots$ for $f$ in $X$ and choose a convergent subsequence.

**Example 2.5.** In particular, any function on $\mathbb{S}^{n-1}$ has a maximum. It follows that any Rayleigh quotient of a symmetric $n \times n$ matrix has a maximum value, which turns out to be its largest eigenvalue. If you had to prove the spectral theorem for symmetric matrices from scratch, Theorem 2.2 is really the most technically difficult step; however, Theorem 2.2 is often used that it becomes second nature most mathematicians.

**Example 2.6.** Similarly any two norms on $\mathbb{R}^n$ are equivalent.

Many fundamental ideas in mathematics amount to maximizing functions over a set in some context, such as continuous functions on a closed and bounded subset of $\mathbb{R}^n$.

2.2. **Open Subsets of $\mathbb{R}^n$.** The notion of a *topological space* is motivated by the fact that the $\delta, \epsilon$ definition of continuity above (i.e., 4(b)) can be expressed more elegantly for maps $f \colon X \to Y$ in the special case where $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$.

**Definition 2.7.** A subset $U \subset \mathbb{R}^n$ is *open* if its complement, i.e., $\mathbb{R}^n \setminus U$, is closed; equivalently, for every $\mathbf{x}_0 \in U$ there exists an $\epsilon > 0$ such that $U$ contains every $\mathbf{x} \in \mathbb{R}^n$ such that $\|\mathbf{x} - \mathbf{x}_0\| < \epsilon$.

**Theorem 2.8.** *A function $f \colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous iff for every open subset $U \subset \mathbb{R}^m$,*

$$f^{-1}(U) \stackrel{def}{=} \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \in U\}$$

*is an open subset of $\mathbb{R}^n$.*

Hence, although (4a) of Subsection 2.1 is the usual "intuitive" notion of continuity, Theorem 2.8 (which is a rewording of (4b)) is an (perhaps the) essential origin of the notion of topological spaces.

2.3. **Topological Spaces and the Induced Topology on $X \subset \mathbb{R}^n$.** We already know what is meant by a continuous map $X \to \mathbb{R}^m$ when $X \subset \mathbb{R}^n$. However, if if $n \geq 2$ and $X = \mathbb{S}^{n-1} \subset \mathbb{R}^n$, then the only subset of $X = \mathbb{S}^{n-1}$ that is an open subset of $\mathbb{R}^n$ is the empty set.

However, the following generalization of Theorem 2.8 is true for maps $X \to \mathbb{R}^m$ with $X \subset \mathbb{R}^n$.

**Theorem 2.9.** *Let $X \subset \mathbb{R}^n$. A function $f \colon X \to \mathbb{R}^m$ is continuous iff for every open subset $U \subset \mathbb{R}^m$,*

$$f^{-1}(U) \stackrel{def}{=} \{\mathbf{x} \in X \mid f(\mathbf{x}) \in U\}$$

*can be written as $X \cap W$, where $W$ is an open subset of $\mathbb{R}^n$.*

This follows from the equivalence of (4a) and (4b) in Subsection 2.1; the main point is that if $S \subset X$ is a subset such that for all $\mathbf{x} \in S$ there is an $\epsilon_{\mathbf{x}} > 0$ such that $\|\mathbf{y} - \mathbf{x}\| < \epsilon_{\mathbf{x}}$ implies that $\mathbf{y} \in S$, then the set

$$W = \bigcup_{\mathbf{x} \in S} \bigl\{\mathbf{y} \in \mathbb{R}^n \ \bigm|\ |\mathbf{y} - \mathbf{x}| < \epsilon_x\bigr\}$$

is an open subset of $\mathbb{R}^n$ (and it is easy to check that $X \cap W = S$). In the next section we will formally prove this, where we conceptualize the above argument by speaking of the *open ball of radius $\epsilon$ about* $\mathbf{x} \in \mathbb{R}^n$,

$$B_{<\epsilon}(\mathbf{x}) \stackrel{\text{def}}{=} \bigl\{\mathbf{y} \in \mathbb{R}^n \ \bigm|\ |\mathbf{y} - \mathbf{x}| < \epsilon\bigr\}$$

and we use the fact that an arbitrary union of open balls — or, more generally, of open sets (Definition 2.7) — in $\mathbb{R}^n$ is again an open set. If you've never seen any point-set topology, this theorem and style of proof may take some "getting used to," and so you may want to move on and wait for a fuller discussion in the next section.

Before proving this theorem, let us restate it more simply.

**Definition 2.10.** A *topological space* is a pair $(X,\mathcal{U})$ consisting of a set $X$ and a family, $\mathcal{U}$, of subsets of $X$ (i.e., the elements of $\mathcal{U}$ are subsets of $X$), that is closed under taking finite intersections and arbitrary unions. Hence $\emptyset, X$ (which are, respectively, the "empty union" and the "empty intersection") must lie in $\mathcal{U}$.[3] We refer to (1) any element of $X$ as a *point* (of the space $(X,\mathcal{U})$) (2) any element of $\mathcal{U}$ as an *open set* (of $(X,\mathcal{U})$), and (3) any set $Z = X \setminus U$ (the complement of $Z$ in $X$) with $U \in \mathcal{U}$ as a *closed subset*.

**Example 2.11.** The pair $(\mathbb{R}^n,\mathcal{U})$ where $\mathcal{U}$ is the subset of open subsets of $\mathbb{R}^n$ (in Definition 2.7) is easily seen to be a topological space. (This is an EXERCISE if you've never verified this.) When we speak of $\mathbb{R}^n$ as a topological space, we always mean this topology, i.e., this notion of an open set, unless we specify otherwise.

**Example 2.12** (The induced topology)**.** If $(X,\mathcal{U})$ is a topological space and $X' \subset X$, then the *induced topology of* $(X,\mathcal{U})$ *on* $X'$ refers to the topological space $(X',\mathcal{U}')$ where $\mathcal{U}' = \{X' \cap U \mid U \in \mathcal{U}\}$; we often call this the *induced topology on* $X'$ *by* $X$ if $\mathcal{U}$ is understood. Hence, if $X \subset \mathbb{R}^n$, then $X$ has its induced topology from $\mathbb{R}^n$.

Theorem 2.9 can be generalized to maps $X \to \mathbb{R}^m$ with $X \subset \mathbb{R}^n$, or even to maps $X \to Y$ with $\mathbb{R}^m$ as follows.

**Theorem 2.13.** *Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, and consider the topologies induced on $X$ and on $Y$. For any $f\colon X \to Y$, the following are equivalent:*

1. *for all sequences $\mathbf{x}_1, \mathbf{x}_2, \ldots$ in $X$ that converge to an $\mathbf{x} \in X$, we have that $f(\mathbf{x}_1), f(\mathbf{x}_2), \ldots$ converges $f(\mathbf{x})$; and*
2. *for any open subset, $U$ of $Y$, $f^{-1}(U)$ is an open subset of $X$.*

**Remark 2.14.** There are many ways of getting interesting topological spaces. If $(X,\mathcal{U})$ and $(X',\mathcal{U}')$ then the product $X \times X'$ becomes a topological space where an open set is the topology *generated by* sets of the form $U \times U'$ with $U \in \mathcal{U}$ and $U' \in \mathcal{U}'$ (where "generated by" is left to the reader (and discussed in the the next section and possibly in class). If $f\colon X' \to X$ is a map of sets and $(X,\mathcal{U})$ is a topological space, then

$$f^{-1}(\mathcal{U}) = \{f^{-1}(U) \mid U \in \mathcal{U}\}$$

generates a topology on $X'$. If $(X,\mathcal{U})$ is a space and $\sim$ is an equivalence relation on $X$, then $X/\sim$ has a topology generated by $U/\sim$ (this is useful for "gluing" spaces together). Etc.

EXERCISE: Show that the topology on $\mathbb{R} \times \mathbb{R}$ above is the same as that on $\mathbb{R}^2$.

EXERCISE: In the above remark we specify that a collection $\mathcal{F}$ of subsets must be open, and then we get then we pass to the topology generated by $\mathcal{F}$. Which of these families $\mathcal{F}$ are already a topology?

EXERCISE: Etc.

## 3. Topological Spaces Arising From Metric Spaces

Many of our topological spaces $(X,\mathcal{U})$ arise when $X$ is endowed with a *metric*; these spaces have a lot of nice properties; they also provide valuable examples and

[3] If you don't like thinking about "empty unions" and "empty intersections," then you can add the condition that $\emptyset, X \in \mathcal{U}$.

intuition in topology (provided that you know of a few topological spaces that aren't metrizable so that misconceptions don't arise...).

Notice that the notion of a convergent sequence i.e., $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$, (i.e., (2) at the beginning of Subsection 2.1) makes sense for points in any set $X$ whenever we know what $|\mathbf{x}_n - \mathbf{x}|$ means.

For this reason we can define convergent sequences in the wide context of *metric space*. In brief, a metric space $(X, \rho)$ is a set, $X$, and a metric or "distance function" $\rho\colon X \times X \to \mathbb{R}_{\geq 0}$ that is symmetric, satisfies the triangle inequality, and $\rho(x, y) = 0$ iff $x = y$. We often write $X$ for $(X, \rho)$ when $\rho$ is understood.

[**Example 1:** $(\mathbb{R}^n, \rho_p)$ where where $\rho_p$ is based on the $L^p$-norm (a.k.a. $\ell^p$-norm) on $\mathbb{R}^n$. **Example 2:** if $X' \subset X$, then $(X', \rho|_{X' \times X'})$ is a metric space. **Example 3:** the *cartesian product* of two metric spaces can be defined in a number of ways, most simply by adding the metrics, more complicatedly by taking applying a norm on $\mathbb{R}^2$ to the pair of metrics.]

**Definition 3.1.** We say that a sequence $x_1, x_2, \ldots$ in a metric space $X = (X, \rho)$ *converges* to an $x \in X$ if as $i \to \infty$, $\rho(x_i, x) \to 0$.

**Definition 3.2.** To any metric space $(X, \rho)$ we associate a topological space $(X, \mathcal{U}_\rho)$ where $U \in \mathcal{U}_\rho$ iff

$$\forall x \in U, \exists \epsilon > 0, \text{ s.t. } \rho(y, x) < \epsilon \Rightarrow y \in U.$$

We say that a topological space $(X, \mathcal{U})$ is *metrizable* if there is a metric on $X$ such that $\mathcal{U} = \mathcal{U}_\rho$.

**Definition 3.3.** Let $(X, \rho)$, $(X', \rho')$ be metric spaces be sets and $f\colon X \to X'$. We say that:

(1) $X$ is *compact* (sometimes *sequentially compact*) if every sequence in $X$ has a convergent subsequence;
(2) $f$ is *continuous* if for every convergence sequence $x_n \to x$ in $X$, we have that $f(x_n) \to f(x)$.

**Definition 3.4.** Let $(X, \mathcal{U})$, $(X', \mathcal{U}')$ be topological spaces be sets and $f\colon X \to X'$. We say that:

(1) $X$ is *compact* (sometimes *topologically compact* for emphasis) if every open covering of $X$ has a finite subcovering;
(2) $f$ is *continuous* if for every open $U'$ in $X'$, $f^{-1}(U')$ is open in $X$.

[In French, our notion of compact is usually called *quasi-compact*, and the term *compact* further assumes that the space is *separated*, i.e., for any distinct points $x, y \in X$, there are disjoint open sets $U_x, U_y$ that respectively contain $x, y$.]

EXERCISE (If you don't know this): Let $(X, \rho)$, $(X', \rho')$ be metric spaces be sets and $f\colon X \to X'$. Let $(X, \mathcal{U}_\rho)$ and $(X', \mathcal{U}_{\rho'})$ be the associated topological spaces. Then $(X, \rho)$ is (sequentially) compact iff $(X, \mathcal{U}_\rho)$ is; and $f$ is continuous as a map of metric spaces iff $f$ is continuous as a map of topological spaces.

**Definition 3.5.** For any $x \in X$ and real $\epsilon > 0$ we define the *open ball of radius* $\epsilon$ *about* $x$ *to be*

$$B_{<\epsilon}(x) = \{y \in X \mid \rho(y, x) < \epsilon\},$$

and the *closed ball of radius* $\epsilon$ *about* $x$ *to be*

$$B_{\leq\epsilon}(x) = \{y \in X \mid \rho(y, x) \leq \epsilon\}$$

(for closed balls we might allow $\epsilon = 0$).

The term "ball" is used to mean an open ball or closed ball depending on the context. Here, in discussing open sets, we will generally mean open balls.[4]

**Definition 3.6.** If $X$ is a set, and $\mathcal{F}$ is a set of subsets of $X$, then the *topology (on $X$) generated by* $\mathcal{F}$ is topological space $(X,\mathcal{U})$ where $\mathcal{U}$ consists of all finite intersections and arbitrary unions of sets in $\mathcal{F}$; equivalently it is the "smallest" topological space on $X$ whose open sets include $\mathcal{F}$.

Here are a bunch of possible additional definitions and EXERCISES. [Some of these would need to be stated more precisely.]

(1) Prove the equivalence of the notions of compactness on a metric space and on its associated topological space.
(2) Prove the equivalence of the notions of a continuous function from a metric space to another, and of continuous as a function of associated topological spaces.
(3) Prove that if $\|\cdot\|$ is any norm on $\mathbb{R}^n$ and we define $\rho(x,y) = \|x-y\|$, then the topology on $\mathbb{R}^n$ induced by $\rho$ is the same as the usual topology.
(4) Let $(X,\rho)$ is a metric space, and for a $y \in X$, let $\rho_y \colon X \to \mathbb{R}$ be the function $\rho_y(x) = \rho(x,y)$. Show that $\rho_y$ is a continuous function.
(5) Let $(X,\mathcal{U})$ be a topological space, and $A \subset X$. The *interior of a* $A$ is the union of all open sets in $X$ that are contained in $A$. The *closure* of $A$, often denoted $\overline{A}$, is the intersection of all closed subsets of $X$ containing $A$. Show that the closure of $X/A$ is the complement in $X$ of the interior of $A$.
(6) A metric space is *complete* if every Cauchy sequence in the space has a limit (such a limit would necessarily be unique). Completeness is an important property, but perhaps not to us this term.
(7) If $(X,\rho)$ is a metric space, and $X' \subset X$, then $X'$ becomes a metric space under the metric $\rho'$ that is $\rho$ restricted to $X' \times X'$. Show that the topology induced on $X'$ from $(X,\mathcal{U}_\rho)$ is the same as the topology induced on $X'$ by $\rho'$.
(8) Prove Theorem 2.13 using some of the facts above.
(9) For any metric space $(X,\rho)$, and any $x \in X$ and $\epsilon > 0$, show that the closure $B_{<\epsilon}(x)$ lies in $B_{\le\epsilon}(x)$. Show that the two are equal if $X$ is *path connected*, i.e., for every $x_0, x_1 \in X$, there is a continuous function $c\colon [0,1] \to X$ such that $c(0) = x_0$ and $c(1) = x_1$. Show that the two are not necessarily equal, even if $X$ consists of only two points.
(10) ADD SOMETHING MORE.
(11) ADD SOMETHING MORE.

## References

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[Fri71] Avner Friedman, *Advanced calculus*, Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1971, Currently published by Dover. MR 352342

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002, Hatcher makes available a free PDF of this textbook. MR 1867354

[4]By contract, in analysis, say when you are finding the eigenvalues of Laplacian on a domain (open, bounded set) in $\mathbb{R}^n$, you would likely want to know that "the unit ball in a separable Hilbert space is weakly compact;" here you are referring to the closed unit ball.

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