(DRAFT:) INTRODUCTION TO SIMPLICIAL HOMOLOGY

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Contents

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Disclaimer: The material may sketchy and/or contain errors, which I will elaborate upon and/or correct in class. For those not in CPSC 531F: use this material at your own risk...

Notes: In class I will often give (extra) examples and draw pictures to clarify and provide intuition for the ideas in this article.

We often use *italics* for precise mathematical terms that we have not (yet) defined; we use "quotation marks" to delimit terms that are vague or whose precise definitions depends on the context and/or author(s). [A sentence or phrase in square brackets is not essential to the rest of the article.]

0. Preface

The first time I gave this course was Spring Term 2025. The purpose of CPSC 531F was to introduce students to the necessary tools of algebraic topology, enough so that they could understand so called "TDA (Topological Data Analysis)."

Of course, there is little sense in rattling off definitions and drawing diagrams if you don't give enough examples so that the audience has a good intuitive sense of what is going on.

ETC.

Part 1. Simplicial Complexes and Abstract Simplicial Complexes

1. Introduction

A lot of TDA (topological data analysis) is based on *simplicial homology*. Some textbooks, notably Munkres' classic *Elements of Algebraic Topology* [Mun84], begin by discussing simplicial complexes and simplicial homology. This is a source of a lot of great examples and intuition. There are, however, numerous potential pitfalls to this approach. Since Munkres' textbook is quite expensive at present, I will also refer to Matoušek's textbook [Ms03] and Armstrong's [Arm83], currently free to download for UBC students.

UBC's very own Prof. Klaus Hoechsmann [well-known for his contribution to helping to hammer out the foundations of the modern (co)homological approach to class field theory [CF67, CF10]], used to teach intro linear algebra courses by teaching the entire course restricted to 2×2 matrices and systems in the first two weeks, and then going back over the entire course, again, this time in the general case. Inspired by Klaus, we will likely do the analog for simplicial complexes, first covering case of (1) 1-dimensional complexes in $\mathbb{R}^N$, and (2) complexes that we can draw in $\mathbb{R}^2$ (hence of dimension at most 2).

2. The Crafty Definition of Simplicial Complexes

In this section we give the standard definition of a simplicial complex. It is "crafty," in the sense that if we change the definition slightly, things tend to go badly, even though this is not obvious at first.

2.1. Simplicies in $\mathbb{R}^N$.

Definition 2.1. A finite sequence of real numbers $(\alpha_0, \ldots, \alpha_d)$ is *stochastic* if $\alpha_i \geq 0$ for all i, and $\alpha_0 + \cdots + \alpha_d = 1$. Let $N \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$, and let $A = \{\mathbf{a}_0, \ldots, \mathbf{a}_d\}$

be a finite subset of $\mathbb{R}^N$. We say that a vector $\mathbf{b} \in \mathbb{R}^N$ is a *convex combination of* $\mathbf{a}_0, \ldots, \mathbf{a}_d$ (or *of* A) if we can write

$$\mathbf{b} = \alpha_0 \mathbf{a}_0 + \cdots + \alpha_d \mathbf{a}_d, \text{ where } (\alpha_0, \ldots, \alpha_d) \text{ is stochastic} \tag{1}$$

The *convex hull of* A is, denoted $\mathrm{conv}(A)$ or $\mathrm{conv}(\alpha_0, \ldots, \alpha_d)$ is the set of all convex combinations in A.

[More generally, the convex hull of an arbitrary set $A \subset \mathbb{R}^N$ is defined as the intersection of all convex sets in $\mathbb{R}^N$ containing A.]

Definition 2.2. Let $N \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$, and let $A = \{\mathbf{a}_0, \ldots, \mathbf{a}_d\}$ be a finite subset of $\mathbb{R}^N$. We say that A is in *general position* (Munkres [Mun84] uses *geometrically independent*, Matoušek [Ms03] uses *affinely independent*) if any of the following equivalent conditions hold:

(1) the vectors $\mathbf{a}_i - \mathbf{a}_0$ with i ranging over $[d] = \{1, 2, \ldots, d\}$ are linearly independent;
(2) the vectors $\mathbf{a}_i - \mathbf{a}_0$ span a d-dimensional subspace of $\mathbb{R}^N$;
(3) the vectors $\mathbf{a}_i - \mathbf{a}_j$ (with $i, j \in [d]$ ranging over $\{0, 1, \ldots, d\}$) span a d-dimensional subspace of $\mathbb{R}^N$;
(4) if M is the $d \times N$ matrix whose i-th row is $\mathbf{a}_i - \mathbf{a}_0$, then M is of rank d;
(5) the vectors $\mathbf{a}_0, \ldots, \mathbf{a}_d$ aren't contained in some $(d-1)$-dimensional *affine linear subspace* of $\mathbb{R}^N$ (i.e., there is no subspace $W \subset \mathbb{R}^N$ of dimension $d-1$ and vector $\mathbf{t} \in \mathbb{R}^N$ such that each $\mathbf{a}_i$ can be written as $\mathbf{t} + \mathbf{w}_i$ for some $\mathbf{w}_i \in W$).

EXERCISE: Any vector $\mathbf{b}$ in the convex hull of a set $A = \{\mathbf{a}_0, \ldots, \mathbf{a}_d\}$ in $\mathbb{R}^N$ can be written as

$$\mathbf{b} = \alpha_0 \mathbf{a}_0 + \cdots + \alpha_d \mathbf{a}_d \quad \text{for some stochastic vector } (\alpha_0, \ldots, \alpha_d). \tag{2}$$

Prove that if $\{\mathbf{a}_0, \ldots, \mathbf{a}_d\}$ are in general position, then for any $\mathbf{b}$ satisfying (2), the vector $(\alpha_0, \ldots, \alpha_d)$ is unique. If so the vector $(\alpha_0, \ldots, \alpha_d)$ is known as the *barycentric coordinates* of $\mathbf{b}$ with respect to $(\mathbf{a}_0, \ldots, \mathbf{a}_d)$ (we write the $\mathbf{a}_i$ in a tuple because their order matters when giving barycentric coordinates.

EXERCISE: Prove that if $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^N$ are any three vectors in generalized position, and if

$$\mathrm{Conv}(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2) = \mathrm{Conv}(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2),$$

then

$$\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2\} = \{\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2\}.$$

Do this by proving that none of $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2$ equals $\mathbf{a}_0$, then $\mathbf{a}_0$ cannot lie in $\mathrm{Conv}(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2)$. [Hint: It may help to express $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2$ in barycentric coordinates.] Then prove the analogous result where $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ is replaced by any arbitrary set $\mathbf{a}_0, \ldots, \mathbf{a}_d \mathbb{R}^N$ in general position.

DRAW PICTURE HERE. Could be a triangle, and its medians, which all meet at the centre of mass of all three vertices.

EXERCISE: This will be needed a bit later: for $i \in \mathbb{N} = \{1, 2, \ldots\}$, let $\mathbf{x}_i = (i, i^2, i^3) \in \mathbb{R}^3$. Show that for any distinct $i, j, k, \ell \in \mathbb{N}$ we have that $\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k, \mathbf{x}_\ell$ are in general position. You may use the fact that any Vandermonde matrix, such as a 4×4 matrix of the form

$$\begin{bmatrix} 1 & a_1 & a_1^2 & a_1^3 \\ 1 & a_2 & a_2^2 & a_2^3 \\ 1 & a_3 & a_3^2 & a_3^3 \\ 1 & a_4 & a_4^2 & a_4^3 \end{bmatrix}$$

with $a_1, \ldots, a_4$ distinct is invertible, i.e., has nonzero determinant.

EXERICSE: Consider the unit circle in $\mathbb{R}^2$, i.e.,

$$\mathbb{S}^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}.$$

Show that any line, $ax_1 + bx_2 = c$, in $\mathbb{R}^2$ intersects $\mathbb{S}^1$ in at most two points. Then show that any three points on $\mathbb{S}^1$ are in general position.

Definition 2.3. Let $d \geq -1$ and $N \geq 0$ be integers. A *d-dimensional simplex in* $\mathbb{R}^N$ (or simply a *d-simplex*) refers to any set, S, that is the convex hull of a subset $A = \{\mathbf{a}_0, \ldots, \mathbf{a}_d\} \subset \mathbb{R}^N$ in general position. If so, then S uniquely determines the set $A = \{\mathbf{a}_0, \ldots, \mathbf{a}_d\}$ (see Exercise **??**), and A is called the *set of vertices* of S. A *face* of S is the convex hull of any subset of A; specifically, a d'-face of S is the convex hull of a subset of A of size $d' + 1$.

Note that a 0-simplex consists of a single point in $\mathbb{R}^N$, and there is only one (-1)-simplex, namely the empty set $\emptyset$. [At times we will ignore $\emptyset$; some authors prefer not to include $\emptyset$ as part of a simplicial complex, allowing only for d-simplicies with $d \geq 0$.]

Remark 2.4. Some authors use the term *face* of a d-dimensional simplex to mean only a face of dimension $d - 1$. Also, when we deal with *simplicial complexes* later, at times some authors use the term *face* to mean a 2-simplex, and *tetrahedron* to mean a 3-simplex.

2.2. **Simiplicial Complexes.** Now we get to the following “crafty” definition.

Definition 2.5. A *simplicial complex* in $\mathbb{R}^N$ is a finite set K whose elements are simplicies in $\mathbb{R}^N$ such that if $S, S' \in K$, then (1) any face of S lies in K, and (2) $S \cap S'$ is a face both of S and of S'. If so, then we use $|K| = |K|_{\mathrm{geom}}$ to denote the union of the elements of K; hence $|K| \subset \mathbb{R}^N$. The *vertex set of* K, denoted $V = V(K)$, refers to the set of all vertices of simplies in K (or, equivalently, all 0-simplicies $\{v\}$ contained in K.

Authors who don't consider $\emptyset$ to be a simplex would write: either $S \cap S'$ is empty, or $S \cap S'$ is both a face of S and of S'.

[DRAW SOME STANDARD PICTURES HERE. OR SEE Figure 2.1 of [Mun84] or page 9 of [Ms03] or Figure 6.3 of [Arm83]]

The reason this definition is “crafty” is that:

(1) if you change this definition slightly, things go badly, or, at least, some subtleties arise;
(2) (at this point) we have no idea why this definition (and not some variant of it) is useful.

Let us elaborate on these two points.

2.3. **Abstract Simplicial Complexes.** A simplicial complex, K, is a collection of simplicies in $\mathbb{R}^N$. It is simpler to keep track of K by knowing: (1) its set of vertices $V = V(K)$ in $\mathbb{R}^N$, and (2) which subsets of V are the vertices of the simplicies in K.

Definition 2.6. Let V be a set (not necessarily a subset of $\mathbb{R}^N$). An *abstract simplicial complex with vertex set* V refers to any set, $\mathsf{K}_{\mathsf{abs}}$, of subsets of V such that (1) $\{v\} \in \mathsf{K}_{\mathsf{abs}}$ for all $v \in V$, and (2) if $A \subset \mathsf{K}_{\mathsf{abs}}$, then $\mathsf{K}_{\mathsf{abs}}$ contains all subsets of A. We define the *dimension* of an element, $A \in \mathsf{K}_{\mathsf{abs}}$ as $|A|-1$, and the *dimension of* $\mathsf{K}_{\mathsf{abs}}$ the largest dimension of an element of $\mathsf{K}_{\mathsf{abs}}$. Unless indicated otherwise, we assume V (and therefore $\mathsf{K}_{\mathsf{abs}}$) is finite.

Example 2.7. Let $G = (V, E)$ be a simple graph; this means that V is a set (finite unless otherwise specified), and E is a collection of subsets of V of size 2 (hence $|E| \le \binom{|V|}{2}$). Then $\mathsf{K}_{\mathsf{abs}} = \{\emptyset\} \cup V \cup E$ (where we understand that V really refers to the sets $\{v\}$ with $v \in V$) is an abstract simplicial complex.

Example 2.8. Let $K \subset \mathbb{R}^N$ be a simplicial complex. Then each $S \in K$ is a simplex of dimension d, which is uniquely determined as the convex hull of a set of vectors $\{\mathbf{a}_0, \ldots, \mathbf{a}_d\}$ in general position. The *abstract simplicial complex* associated to K is the simplicial complex whose vertex set consists of the 0-simplices of K (i.e., the one-element sets in K) and all subsets $\{\mathbf{a}_0, \ldots, \mathbf{a}_d\}$ of V such that $S = \operatorname{conv}(\mathbf{a}_0, \ldots, \mathbf{a}_d)$ is an element of K.

Remark 2.9. We warn the reader to remember that a simplicial complex, K, is a set whose elements are simplicies, S, in $\mathbb{R}^N$ for some N. Their union $|K| = |K|_{\text{geom}}$ is a subset of $\mathbb{R}^N$. To K we associate its vertex set, $V \subset \mathbb{R}^N$, and an abstract simplicial complex, $\mathsf{K}_{\mathsf{abs}}$, whose elements are subsets of V. Note that knowing K is equivalent to knowning $|K|$, and this is equivalent to knowing $\mathsf{K}_{\mathsf{abs}}$.

2.4. **Craftiness 1: Why Do Topologists Care About Simplicial Complexes?** Algebraic topology concerns itself with *topological spaces*. For any topological space, X, one can define its homology groups $H_0(X), H_1(X), \ldots$, as well as its homotopy groups (which we will not focus on). If $\mathsf{K}_{\mathsf{abs}}$ is any abstract simplicial complex, one can define its *simplicial homology groups*, $H_i^{\text{simp}}(\mathsf{K}_{\mathsf{abs}})$, that can be computed with finite dimensional linear algebra arising from the combinatorics of $\mathsf{K}_{\mathsf{abs}}$. If K is a simplicial complex and $\mathsf{K}_{\mathsf{abs}}$ its associated simplicial complex, then $|K| = |K|_{\text{geom}} \subset \mathbb{R}^N$, which therefore becomes a topological space, and it turns out (after a fairly long proof...) that $H_i^{\text{simp}}(\mathsf{K}_{\mathsf{abs}}) = H_i(|K|)$. Hence, to a topologist, if one has a topological space, X, that is isomorphic (or even merely homotopic) to $|K|$ for some simplicial complex, K, then we get a tool for computing $H_i(X)$.

Working with topological spaces is in many ways much simpler than working with simplicial complexes; we'll see this when defining the product of two topological spaces as opposed to the product of two simplicial complexes, and when proving that two homotopic spaces have isomorphic homology groups. Hence we often want to think in terms of topological spaces and the singular homology groups $H_i(X)$, even if our primary interest is simplicial complexes.

2.5. **Craftiness 2: Can We Require Only that $S \cap S' \in K$?** It may seem that one gets a reasonable notion of simplicial complex if one weakens the condition that $S \cap S'$ is a face of both S and S' to the condition that $S \cap S' \in K$. Here is a problem.

Say that we define a "pseudo simplicial complex" (or whatever term you like...) to be a set, K, of simplicies in $\mathbb{R}^N$ such that (1) any face of an $S \in K$ is again in K, and (2) for any $S, S' \in K$, we have $S \cap S' \in K$. If so, we can still associate to K its union of elements $|K|$, and an abstract simplicial complex, $\mathsf{K}_{\mathsf{abs}}$, of subsets of the vertices whose convex hull is an element of K. In this case, it turns out that $H_1(|K|)$ and $H_1^{\mathrm{simp}}(\mathsf{K}_{\mathsf{abs}})$ are no longer equal. Let's give an example.

For real a, b with $a \le b$, let $[a, b] \subset \mathbb{R}$ denote (as usual) the closed interval $\{x \mid a \le x \le b\}$.

$$K^{\mathrm{bad}} = \bigl\{\emptyset, \{0\}, \{1\}, \{2\}, [0,2], [0,1], [1,2]\bigr\},$$

which is not a simplicial complex, since $[0,2] \cap [1,2] = [1,2]$ which is not a face of $[0,2]$. However, it does satisfy the condition that $S \cap S' \in K^{\mathrm{bad}}$ whenever $S, S' \in K^{\mathrm{bad}}$. If we look at the sets of vertices of the elements in K^{bad}, we get an abstract simplicial complex:

$$\mathsf{K}_{\mathsf{abs}}^{\mathrm{bad}} = \bigl\{\emptyset, \{0\}, \{1\}, \{2\}, \{0,2\}, \{0,1\}, \{1,2\}\bigr\}.$$

However, $\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^2$ are any three vectors in general position, then the abstract simplicial complex

$$\mathsf{K}_{\mathsf{abs}} = \bigl\{\emptyset, \{\mathbf{s}_0\}, \{\mathbf{s}_1\}, \{\mathbf{s}_2\}, \{\mathbf{s}_0, \mathbf{s}_2\}, \{\mathbf{s}_0, \mathbf{s}_1\}, \{\mathbf{s}_1, \mathbf{s}_2\}\bigr\}$$

is combinatorially the same abstract simplicial complex, after renaming the vertex i of $\mathsf{K}_{\mathsf{abs}}^{\mathrm{bad}}$ to $\mathbf{s}_i$ in $\mathsf{K}_{\mathsf{abs}}$. However, $\mathsf{K}_{\mathsf{abs}}$ has its vertices in $\mathbb{R}^2$, and the associated simplicial complex is

$$K = \bigl\{\emptyset, \{\mathbf{s}_0\}, \{\mathbf{s}_1\}, \{\mathbf{s}_2\}, \mathrm{conv}(\mathbf{s}_0, \mathbf{s}_2), \mathrm{conv}(\mathbf{s}_0, \mathbf{s}_1), \mathrm{conv}(\mathbf{s}_1, \mathbf{s}_2)\bigr\},$$

so $|K|_{\mathrm{geom}}$ is a triangle (without the interior) in $\mathbb{R}^2$. $\mathsf{K}_{\mathsf{abs}}$ is also a graph, and as a graph it has exactly one *cycle*. We will see that if an abstract complex is a graph with a single cycle, then its H_1^{simp} is one-dimensional, and similarly for one-dimensional topological spaces in $\mathbb{R}^N$ and H_1 (under mild restictions, understanding "cycle" to be its intuitive meaning). Hence we will have

$$H_1^{\mathrm{simp}}(\mathsf{K}_{\mathsf{abs}}^{\mathrm{bad}}) \simeq H_1^{\mathrm{simp}}(\mathsf{K}_{\mathsf{abs}}) \simeq H_1(|K|) \simeq \mathbb{R};$$

and by contrast, since $|K^{\mathrm{bad}}|$ is the interval $[0,2]$, which has no "cycles," it will turn out that

$$H_1(|K^{\mathrm{bad}}|) \simeq \mathbb{R}^0 = \{0\}.$$

So when defining simplicial complexes, replacing the condition that "$S \cap S'$ is a face of both S, S'" with "$S \cap S' \in K$" allows for degenerate situations where the value of H_1 is the "wrong" value, i.e., does not agree with simplicial homology.

2.6. **Craftiness 3: Why Insist that K be Finite?** Say that in Definition 2.1 we allow K to be infinite. Then the following sets

$$A = \bigl\{\{P\} \bigm| P \in \{0, 1, 1/2, 1/3, 1/4, \ldots\}\bigr\}, \quad B = \bigl\{\{Q\} \bigm| Q \in \{0, 1, 2, 3, 4, \ldots\}\bigr\}$$

are simplicial complexes, each consisting of a countably infinite set of vertices in $\mathbb{R}$, with no 1-simplicies joining the vertices. Hence it seems like A and B should be the "same complex," and this intuition is correct.

However, if $X, Y \subset \mathbb{R}$, we usually define a function $f\colon X \to Y$ to be *continuous* iff

$$\forall x_0 \in X, \quad \lim_{x \to x_0} f(x) = f(x_0) \tag{3}$$

(of course, have to make this precise...). [You have probably seen this definition if $X = Y = \mathbb{R}$, i.e., a function $f\colon \mathbb{R} \to \mathbb{R}$, but this turns out to be the "correct" definition when $X, Y \subset \mathbb{R}$ where X and Y are understood as "embedded (topological) spaces" $\mathbb{R}$.]

But the function $f\colon |A| \to |B|$ given by $f(0) = 0$ and $f(1/n) = n$ for $n \in \mathbb{N}$ fails to satisfy (3), since as $n \to \infty$, $1/n \to 0$, but $f(1/n) = n$ does not tend to 0 as $n \to \infty$.

Hence the *topologies* (or *metrics*) we put on $|A|$ and $|B|$, i.e., the sense in which (3) is interpreted, will not be the sense as if we view both $|A|, |B|$ as subsets of $\mathbb{R}$.

By contrast, if K, L are *finite* simplicial complexes, with $|K| \subset \mathbb{R}^N$ and $|L| \subset \mathbb{R}^M$, then (3) is the "correct" condition to say that a function $f\colon |K| \to |L|$ is continuous. We'll return to this when we review some point-set topology a bit later.

Remark 2.10. Although in TDA we may always work with simplicial complexes K that are finite, we don't have that luxury in topology. Indeed, $\mathbb{R}$ and the open interval $(0, 1)$, can never be isomorphic as topological spaces (or *homeomorphic*) to $|K|$ for a finite simplicial complex, K, since in this case $|K|$ (with its "reasonable" topology) will be *compact*, and $\mathbb{R}$ and $(0, 1)$ are (isomorphic as topological spaces and) both non-compact.

3. Graphs and Simplicial Complexes

Definition 3.1. Let $K \subset \mathbb{R}^N$ be a simplicial complex. For each $n \in \mathbb{Z}_{\geq 0}$, we define

$$K_n \stackrel{\text{def}}{=} \{S \in K \mid \dim(S) \leq n\},$$

where $\dim(S)$ denotes the dimension of S. We say that K is *of dimension n* if n is the smallest integer such that $K_n = K$. Recall that $V = V(K) \subset \mathbb{R}^N$ is the vertex set of K. Each simplex of dimension 1 in K is a simplex on two points $\{v, v'\}$, with $v, v' \in V$; we refer to the set of all such pairs $\{v, v'\}$ as the *edge set* of K, denoting it by $E = E(K)$. We call $G = (V, E)$ the *graph associated to K*.

It follows that G above is equal to the abstract simplicial complex associated to K_1, i.e., the simplicial complex we get by discarding all simplices of dimension 2 or more.

[DRAW SOME EXAMPLES. For example, the complete graph on 4 vertices can be drawn in the plane, with three vertices in general position and a fourth in the interior of the convex hull of the three vertices.]

A *complete graph on 5 vertices* refers to any graph $G = (V, E)$ where $|V| = 5$ and E consists of all pairs of distinct vertices (hence $|E| = \binom{5}{2}$). [We similarly define the complete graph on any number of vertices.] It is well-known that the complete graph on 5 vertices is not *planar* (in class we will explain what this means; we won't prove this theorem). Let us state a corollary.

Theorem 3.2. *Let K be a simplicial complex in $\mathbb{R}^2$. Then the graph associated to K cannot be a complete graph on 5 vertices.*

However, we do have the following theorem. Recall that two simple graphs, $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there is a bijection $\phi\colon V \to V'$ that induces an isomorphism from E to E' in the sense that for all $v_1, v_2 \in V$, $\{v_1, v_2\} \in E$ iff $\{\phi(v_1), \phi(v_2)\} \in E'$.

Theorem 3.3. *Let $G = (V, E)$ be any graph. Then there is a simplicial complex, K, in $\mathbb{R}^3$, whose associated abstract simplicial complex is isomorphic to G.*

Proof. Let $V = \{v_1, \ldots, v_n\}$, and let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^3$. Let K be the set of simplicies consisting of (1) $\emptyset$, (2) $\{\mathbf{x}_i\}$ for each i, and (3) the 1-simplicies with vertices $\mathbf{x}_i, \mathbf{x}_j$ for all those i, j such that $\{v_i, v_j\} \in E$.

We claim that we can find $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^3$ so that K is a simplicial complex: we need to check that for all $S, S' \in K$ we that that (1) all faces of S lie in K, and (2) if $S \cap S'$ is non-empty, then $S \cap S'$ is a face of both S and S'. Condition (1) holds automatically; condition (2) holds easily unless both S, S' are 1-simplexes. So let S be the 1-simplex with vertices $\mathbf{x}_i, \mathbf{x}_j$, and let S' that with $\mathbf{x}_k, \mathbf{x}_\ell$; note that $i \neq j$ and $k \neq \ell$. We claim that It suffices to check the following two claims:

1. if i, j, k, ℓ are all distinct, then $S \cap S' = \emptyset$; and
2. if $j = k$, then $S \cap S' = \mathbf{x}_j = \mathbf{x}_k$.

(all other cases reduce to one of these two, or the case where $\{i, j\} = \{k, \ell\}$, in which case $S = S'$ and there is nothing to check).

If i, j, k, ℓ are all distinct, we need to make sure that the equation

$$\alpha\mathbf{x}_i + (1-\alpha)\mathbf{x}_j = \beta\mathbf{x}_k + (1-\beta)\mathbf{x}_\ell \tag{4}$$

has no solutions with $\alpha, \beta \geq 0$; so consider any solution to the above equation.

EXERCISE: Show that this implies that the vectors $\mathbf{x}_j - \mathbf{x}_i$, $\mathbf{x}_k - \mathbf{x}_i$, and $\mathbf{x}_\ell - \mathbf{x}_i$ are linearly dependent, i.e., the span a subspace of dimension two or less.

Hence this case cannot occur if:

$$\forall i, j, k, \ell \text{ distinct, } \mathbf{x}_j - \mathbf{x}_i,\ \mathbf{x}_k - \mathbf{x}_i,\ \mathbf{x}_\ell - \mathbf{x}_i \text{ are independent.} \tag{5}$$

Next, if $i, j = k, \ell$ are otherwise distinct, we want to make sure that (4) has a unique solution for $\alpha = 1$ and $\beta = 0$, because if so then both sides of this equation equal $\mathbf{x}_j = \mathbf{x}_k$. EXERCISE: show that if (4) has a solution other than $\alpha = 1$ and $\beta = 0$, then $\mathbf{x}_j - \mathbf{x}_i$ and $\mathbf{x}_\ell - \mathbf{x}_i$ are linearly dependent (i.e., are colinear). Hence this cannot hold if

$$\forall i, k, \ell \text{ distinct, } \mathbf{x}_k - \mathbf{x}_i,\ \mathbf{x}_\ell - \mathbf{x}_i \text{ are independent.} \tag{6}$$

Of course, if $n = |V| \geq 4$, (6) is implied by (5) since if $\mathbf{x}_k - \mathbf{x}_i$, $\mathbf{x}_\ell - \mathbf{x}_i$ are already linearly dependent for some i, k, ℓ distinct, then taking any j distinct from i, k, ℓ we have that (5) does not hold.

For each i let $\mathbf{x}_i = (i, i^2, i^3) \in \mathbb{R}^3$. It is well known that any Vandermonde matrix, such as a 4×4 matrix of the form

$$\begin{bmatrix} 1 & a_1 & a_1^2 & a_1^3 \\ 1 & a_2 & a_2^2 & a_2^3 \\ 1 & a_3 & a_3^2 & a_3^3 \\ 1 & a_4 & a_4^2 & a_4^3 \end{bmatrix}$$

with $a_1, \ldots, a_4$ distinct is invertible, i.e., has nonzero determinant. EXERCISE: Use this fact about Vandermonde matrices to show that the choice of $\mathbf{x}_i$ as given satisfies (5) and (6). $\square$

EXERCISE: Let $\mathsf{K}_{\mathsf{abs}}$ be an abstract simplicial complex of dimension at most 2, i.e., each set in $\mathsf{K}_{\mathsf{abs}}$ has at most 3 elements. Show that there is a simplicial complex $S \subset \mathbb{R}^5$ whose associated abstract simplicial complex is $\mathsf{K}_{\mathsf{abs}}$.

EXERCISE: Let $\mathsf{K}_{\mathsf{abs}}$ be an abstract simplicial complex of dimension at most d, i.e., each set in $\mathsf{K}_{\mathsf{abs}}$ has at most $d+1$ elements. Show that there is a simplicial complex $S \subset \mathbb{R}^{2d+1}$ whose associated abstract simplicial complex is $\mathsf{K}_{\mathsf{abs}}$.

Part 2. Simplicial Homology of Abstract Simplicial Complexes

4. Simplicial Homology Groups

In this section we introduce the simplicial homlogy groups of abstract simplicial complexes. We will focus on simplicial complexes of graphs and of complexes that we can "draw" in $\mathbb{R}^2$.

4.1. Simplicial Homology Groups of Graphs.

Let $G = (V, E)$ be a simple graph. We will define the homology groups, $H_0(G)$ and $H_1(G)$ as $\mathbb{R}$-vector spaces.

4.1.1. *0-forms and formal linear combinations.* First, by a *0-form* or a *0-dimensional chain* on a $G = (V, E)$ we mean the set $\mathcal{C}_0(G)$ consisting of "$\mathbb{R}$-linear formal sums"

$$\sum_{i=1}^{r} \alpha_i v_i, \tag{7}$$

where $\alpha_i \in \mathbb{R}$ and $v_i \in V$.

For the reader new to this idea, let us explain the idea. First let us give the usual definition, although it is a bit imprecise; you can find a precise (and tedious) definition in Definition D.1 in Appendix D.

Definition 4.1. Let S be any set. A *formal $\mathbb{R}$-linear sum in S* refers to a formal sum

$$\alpha_1 s_1 + \cdots + \alpha_r s_r, \tag{8}$$

where we identity two formal sums, writing

$$\alpha_1 s_1 + \cdots + \alpha_r s_r = \alpha'_1 s'_1 + \cdots + \alpha'_{r'} s'_{r'}, \tag{9}$$

if for each $s \in S$, the sum of the α_i over those i with $s_i = v$ equals the sum of the $\alpha'_{i'}$ over those i' with $s'_{i'} = s$ (clearly $=$ in (9) is an equivalence relation). We use $\mathbb{R}[S]$ to denote the set of all such formal sums; $\mathbb{R}[S]$ becomes a vector space under the evident operations of $+$ and scalar multiplication.

Hence the elements of S are a basis for $\mathbb{R}[S]$.

Hence $\mathcal{C}_0(G)$ of a graph $G = (V, E)$ is, by definition, $\mathbb{R}[V]$

For example, if $V = \{P, Q, R\}$, then $\mathbb{R}[V]$ includes

$$2P,\ 3P + (1.7)R\, 2P + 3P - 8P + \sqrt{7}R$$

and if an element of $v \in V$ does not appear in a formal sum, then this is the same as $(0)v$ appearing in the sum; hence

$$(-2)P + (3.2)Q,\ -2P + (3.2)Q + 3R + (-3)R,\ (16Q - 10P)/5$$

are formal sums that are equal.

One can also view a 0-form as a function $V \to \mathbb{R}$, namely for any $f \colon V \to \mathbb{R}$, one associates the formal sum $\sum_{v \in V} f(v)v$.[1] [If V is infinite, one also has to insist that

[1]However, it is better to consider $\mathbb{R}[V]$ and functions $V \to \mathbb{R}$ as *dual* vector spaces, not the same vector space.

this function is zero at all but finitely many values of V.] However, it is usually more convenient to work with formal sums $\mathbb{R}[V]$, since often our formal sums only involve a small number of vertices. Later we will see another advantage, namely if $S \subset T$ are sets, then one can naturally view $\mathbb{R}[S] \subset \mathbb{R}[T]$.

4.2. **1-forms.** Next, by a 1-*form* on G we mean a formal sum

$$\sum_{i=1}^{r} \alpha_i [v, v']$$

of ordered pairs $[v, v']$ such that (1) $\{v, v'\} \in E$, and (2) we understand that $[v, v'] = -[v', v]$. Equivalently, for each edge $e = \{v, v'\} \in E$ we choose an *orientation of* $e = \{v, v'\}$, meaning one ordered pair (v, v'); then we consider formal sums of oriented edges $[v, v']$, but we understand that $[v', v]$ is a synonmym (or shorthand) for $-[v, v']$.

Remark 4.2. Some authors use (v, v') instead of $[v, v']$. More on this below.

4.3. **The boundary map ∂_1 and homology groups.** We use $\mathcal{C}_0 = \mathcal{C}_0(G)$ to denote the set of 0-forms of G, $\mathcal{C}_1 = \mathcal{C}(G$ to denote the set of 1-forms on G, and we define a linear transformation

$$\partial_1 \colon \mathcal{C}_1 \to \mathcal{C}_0$$

to be the unique linear map taking $[v, v']$ to $v' - v$. We then define:

$$H_1^{\rm simp}(G) \stackrel{\rm def}{=} \ker(\partial_1) = \{\mu \in \mathcal{C}_1 \,|\, \partial_1(\mu) = 0\}, \quad H_0^{\rm simp}(G) \stackrel{\rm def}{=} \operatorname{coker}(\partial_1) = \mathcal{C}_0/\mathrm{Image}(\partial_1) = \mathcal{C}_0/\partial_1(\mathcal{C}_1).$$

We also define the *Betti numbers* of G to be

$$\beta_i(G) \stackrel{\rm def}{=} \dim\bigl(H_i^{\rm simp}(G)\bigr).$$

Remark 4.3. If $V = \{v_1, \ldots, v_n\}$ are the vertices of G, then we can always orient the edge in "increasing order," i.e., if $i < j$, we orient $\{v_i, v_j\}$ as (v_i, v_j). However, it is important to see that any closed walk in the graph gives you an element in the kernel of ∂_1, but this is only true if you understand that (v_j, v_i) refers to $-(v_i, v_j)$. Here we use brackets, i.e., $[v_i, v_j]$ and $[v_j, v_i] = -[v_i, v_j]$ to remind us of this.

4.3.1. *An example (which should explain why we want some theorems to compute homology groups and Betti numbers).*

Example 4.4. Let $G = (V, E)$, where $V = \{A, B, C, D\}$ and E consists of all two element subsets of V; in graph theory we say that G is a *complete graph on four vertices*. One typically denotes this by K_4, although one has to understand that this is not a single graph, but an isomorphism class of graphs (e.g., there is also a complete graph on vertex set $V = \{1, 2, 3, 4\}$ or $V = \{\alpha, \beta, \gamma, \delta\}$). Hence a basis for $\mathcal{C}_0(G)$ is given by

$$[A], [B], [C], [D], \tag{10}$$

and a basis for $\mathcal{C}_1(G)$ is given by

$$[A, B], [A, C], [A, D], [B, C], [B, D], [C, D]. \tag{11}$$

Since $\partial_1([A,B]) = [B] - [A]$, the table of this vector's coefficients

	$[A,B]$
$[A]$	-1
$[B]$	1
$[C]$	0
$[D]$	0

shows that with respect to the basis (10)

$$\partial_1([A,B]) = [B] - [A] = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}_{\{[A],[B],[C],[D]\}},$$

where the subscript on the vector indicates the basis elements in (10) and their order. Similarly the full table

	$[A,B]$	$[A,C]$	$[A,D]$	$[B,C]$	$[B,D]$	$[C,D]$
$[A]$	-1	-1	-1	0	0	0
$[B]$	1	0	0	-1	-1	0
$[C]$	0	1	0	1	0	-1
$[D]$	0	0	1	0	1	1

shows us that with respect to the bases (11) and (10), ∂_1 is represented by the matrix

$$M = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \tag{12}$$

By definition, $H_1(G) = \ker(\partial_1)$ we solve the equations

$$\begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

whose solutions $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_6)$ are precisely those $\boldsymbol{\alpha}$ such that

$$\alpha_1[A,B] + \alpha_2[A,C] + \alpha_3[A,D] + \alpha_4[B,C] + \alpha_5[B,D] + \alpha_6[C,D] \in \ker(\partial_1)$$

After a tedious Gaussian elimination[2] we see that M in (12) has the reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[2]Alternatively, in class we'll avoid Gaussian elimination and just use some ad hoc row operations — can you see which ones?

and it follows that $\dim(\ker(M)) = 3$ and $\operatorname{rank}(M) = 6 - \dim(\ker(M)) = 3$; moreover, $\ker(M)$ contains those α of the whose general form is given by

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_4 + \alpha_5 \\ -\alpha_4 + \alpha_6 \\ -\alpha_5 - \alpha_6 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix},$$

and therefore

$$H_1^{\mathrm{simp}}(G) \stackrel{\mathrm{def}}{=} \ker(\partial_1)$$
$$= \Bigl\{(\alpha_4+\alpha_5)[A,B]+(-\alpha_4+\alpha_6)[A,C]+(-\alpha_5-\alpha_6)[A,D]+\alpha_4[B,C]+\alpha_5[B,D]+\alpha_6[C,D] \;\Bigm|\; \alpha_4,\alpha_5,\alpha_6 \in \mathbb{R}\Bigr\}.$$

Since $\operatorname{rank}(M) = 3$, we have $\dim(\operatorname{Image}(M)) = 3$ and $\dim(\operatorname{coker}(M)) = 4 - \dim(\operatorname{Image}(M)) = 1$. It follows that

$$H_0^{\mathrm{simp}}(G) \simeq \mathbb{R}.$$

It is important to remember that

$$H_0^{\mathrm{simp}}(G) \stackrel{\mathrm{def}}{=} \operatorname{coker}(\partial_1) \stackrel{\mathrm{def}}{=} \mathcal{C}_0/\operatorname{Image}(\partial_1)$$

is the *quotient space* $\mathcal{C}_0/\operatorname{Image}(\partial_1)$. Since the column reduced form of M in (12) are linearly independent is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

one can describe the image of ∂_1 as

$$\operatorname{Image}(\partial_1) = \Bigl\{\beta_1[A] + \beta_2[B] + \beta_3[C] + (-\beta_1 - \beta_2 - \beta_3)[D] \;\Bigm|\; \beta_1,\beta_2,\beta_3 \in \mathbb{R}\Bigr\},$$

and

$$\operatorname{coker}(\partial_1) = \bigl(\mathbb{R}[A] + \mathbb{R}[B] + \mathbb{R}[C] + \mathbb{R}[D]\bigr)/\operatorname{Image}(\partial_1).$$

We will elaborate on this in Subsection 4.7, and review the notion of quotient spaces in Subsubsection 4.7.3. Note in the above that the Betti numbers of G,

$$\beta_i(G) \stackrel{\mathrm{def}}{=} \dim\bigl(H_i^{\mathrm{simp}}(G)\bigr),$$

are given $\beta_1 = 3$ and

$$\beta_0 = \dim\bigl(\mathcal{C}_0/\operatorname{Image}(\partial_1)\bigr) = \dim(\mathcal{C}_0) - \dim\bigl(\operatorname{Image}(\partial_1)\bigr) = 4 - \operatorname{rank}(M) = 4 - 3 = 1.$$

Note also that the classical *Euler characteristic of* G, defined as $\chi(G) = |V| - |E|$ satisfies

$$\chi(G) = |V| - |E| = 4 - 6 = -2, \quad \text{and} \quad \beta_0 - \beta_1 = -2.$$

This is not a coincidence, since $\partial_1 : \mathcal{C}_1(G) \to \mathcal{C}_0(G)$ is a map from a $|E|$-dimensional space to a $|V|$-dimensional space, and hence, from general facts in linear algebra,

$$\beta_0 - \beta_1 = \dim(\operatorname{coker}(\partial_1)) - \dim(\ker(\partial_1)) = \bigl(|V| - \operatorname{rank}(\partial_1)\bigr) - \bigl(|E| - \operatorname{rank}(\partial_1)\bigr) = |V| - |E| = \chi(G).$$

Hence, if we know that $\beta_0 = 1$, then we can determine β_1, which determines the groups $H_i^{\mathrm{simp}}(G)$ up to isomorphism (as $\mathbb{R}$-vector spaces).

4.3.2. *Main result on $\beta_0(G)$.* In Subsection 4.7 we will prove the following theorem.

Theorem 4.5. *Let G be a graph. Then:*

(1) *$\beta_0(G)$ is the number of connected components of G.*
(2) *If $\chi(G) = |V| - |E|$ is the usual* Euler characteristic *of G, then $\beta_0(G) - \beta_1(G) = \chi(G)$.*
(3) *$\beta_1(G)$ is the minimum number of edges that we need to remove from G to get a forest.*

If you draw G in a plane (if you can, i.e., if G is a planar graph), then $\beta_1(G)$ should look like the "number of independent cycles" or "number of holes" or "number of polygons" in the graph.

[DRAW SOME EXAMPLES]

Any oriented, closed walk in the graph gives you an element of the kernel of ∂_1.

4.4. **Simplicial Homology for 2-Dimension Complexes.** The problem with looking just at the graph case is that it is unclear what to expect for higher-dimensional simplicial complexes. This becomes clearer if we look at 2-dimensional complexes.

Let $\mathsf{K}_{\mathsf{abs}}$ be an abstract simplicial complex of dimension two. In this case, we have

$$\mathsf{K}_{\mathsf{abs}} = \{\emptyset\} \cup V \cup E \cup F$$

(where we understand that V really refers to the sets $\{v\}$ with $v \in V$ and) where F is the set of 2-faces, i.e., of size 3. We then define a 2-form of $\mathsf{K}_{\mathsf{abs}}$ to be a formal sum

$$\sum_{i=1}^{r} \alpha_i [v_0, v_1, v_2]$$

where $\{v_0, v_1, v_2\} \in F$, and we refer to $[v_0, v_1, v_2]$ as an *orientaiton* of $\{v_0, v_1, v_2\}$, and we understand that

$$[v_0, v_1, v_2] = [v_1, v_2, v_0] = [v_2, v_0, v_1],$$

and that their "reverse orientations" are all equal

$$[v_1, v_0, v_2] = [v_0, v_2, v_1] = [v_2, v_1, v_0]$$

and equal -1 times $[v_0, v_1, v_2]$. DRAW PICTURE.

We then set $\mathcal{C}_2$ to be the vector space of 2-forms, and we define $\partial_2 \colon \mathcal{C}_2 \to \mathcal{C}_1$ to be the uniquel linear map with

$$\partial_2[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1],$$

which we also write as:

$$[\widehat{v_0}, v_1, v_2] - [v_0, \widehat{v_1}, v_2] + [v_0, v_1, \widehat{v_0}]$$

where the notation $\widehat{v_i}$ means "omit v_i." We easily check that ∂_2 is well-defined over 2-forms, i.e., that

$$\partial_2[v_0, v_1, v_2] = \partial_2[v_1, v_2, v_0] = \partial_2[v_2, v_0, v_1] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$
$$= -\partial_2[v_1, v_0, v_2] = -\partial_2[v_0, v_2, v_1] = -\partial_2[v_2, v_1, v_0]$$

DRAW PICTURE.

Now we get to the dramatic calculation that gives rise to homology groups: $\partial_1 \circ \partial_2 : \mathcal{C}_2 \to \mathcal{C}_0 = 0$ (!). This gives us a sequence of vector spaces with connecting maps:

$$0 \to \mathcal{C}_2 \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \to 0$$

such that the composition of any two maps is zero.

Definition 4.6. By a *chain complex*

$$\cdots \xrightarrow{\partial_3} \mathcal{C}_2 \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \xrightarrow{\partial_0} \mathcal{C}_{-1} \xrightarrow{\partial_{-1}} \cdots \tag{13}$$

we mean a sequence of $\mathbb{R}$-vector spaces, $\ldots, \mathcal{C}_{-1}, \mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots$ plus maps $\partial_i : \mathcal{C}_i \to \mathcal{C}_{i-1}$ for all i such that for all i, $\partial_i \partial_{i+1} = 0$. We often use $(\mathcal{C}_\bullet, \boldsymbol{\partial}_\bullet)$ or just $(\mathcal{C}, \boldsymbol{\partial})$ to denote this chain, or even $\mathcal{C}$ (with $\boldsymbol{\partial}$ understood). The *i-th homology group of this chain complex* is the $\mathbb{R}$-vector space

$$H_i^{\mathrm{simp}} = H_i^{\mathrm{simp}}(\mathcal{C}) = \ker(\partial_i)/\mathrm{Image}(\partial_{i+1}).$$

The point is that because $\partial_i \partial_{i+1} = 0$, it follows that the image of ∂_{i+1} is a subspace $\ker(\partial_i)$. It is common to define the *i-cycles* of this chain to be the kernel of ∂_i, and to denote them by Z_i (*Zyklus* in German), and the *i-boundaries* of this chain to be the image of ∂_{i+1}, denoted B_i. Then $B_i \subset Z_i$ and the homology groups become:

$$H_i^{\mathrm{simp}}(\mathcal{C}) = Z_i(\mathcal{C})/B_i(\mathcal{C}).$$

Drawing some pictures with graphs and 2-dimensional simplicial complexes explains this terminology, and provides the intuition.

In practice, for simplicial homology of an abstract simplicial complex, $\mathsf{K}_{\mathsf{abs}}$, we typically have $\mathcal{C}_i = 0$ for $i \le -1$ (or $i \le -2$ when we define *reduced homology*), and $\mathcal{C}_i = 0$ for $i > \dim(\mathsf{K}_{\mathsf{abs}})$. Hence the (conceivably "doubly-infinite") chain complex (13) is really a finite chain complex with a 0 vector space at either end.

In particular, for a graph $G = (V, E)$ we have at most a single non-trivial map $\mathcal{C}_0$, and the homology groups of

$$0 \to \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \to 0$$

are a somewhat "degenerate" case where:

$$H_1^{\mathrm{simp}} = Z_1/B_1 = Z_1 = \ker(\partial_1), \quad H_0^{\mathrm{simp}} = Z_0/B_0 = \mathcal{C}_0/B_0 = \mathrm{coker}(\partial_1).$$

Whereas, for a 2-dimensional abstract simplicial complex, $\mathsf{K}_{\mathsf{abs}}$, we get a chain complex

$$0 \to \mathcal{C}_2 \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \to 0$$

whereupon

$$H_2^{\mathrm{simp}} = Z_2 = \ker(\partial_2), \quad H_1^{\mathrm{simp}} = Z_1/B_1 = \ker(\partial_1)/\mathrm{Image}(\partial_2), \quad H_0^{\mathrm{simp}} = \mathcal{C}_0/B_0 = \mathrm{coker}(\partial_0).$$

Let us give some examples. To simplify notation, we introduce the following definition.

Definition 4.7. For a set V and subsets $S_1, \ldots, S_r \subset V$, the *abstract simplicial complex on vertex set V generated by $S_1, \ldots, S_r$* is the smallest abstract simplicial complex complex on vertex set V containing $S_1, \ldots, S_r$, i.e.,

$$\langle S_1, \ldots, S_r \rangle_V = V_{\mathrm{single}} \cup \mathrm{Power}(S_1) \cup \ldots \cup \mathrm{Power}(S_r),$$

where $\mathrm{Power}(S)$ denotes the power set of S (i.e., the set of all subsets of S), and V_{single} denotes the set consisting of $\emptyset$ and singletons (i.e., 1-element subsets) of V. Similarly, the *abstract simplicial complex generated by* $S_1, \ldots, S_r$ is

$$\langle S_1, \ldots, S_r \rangle = \mathrm{Power}(S_1) \cup \ldots \cup \mathrm{Power}(S_r)$$

(whose vertex set is therefore $S_1 \cup \ldots \cup S_r$).

Example 4.8. Let $\mathsf{K}_{\mathsf{abs}} = \langle \{A, B, C\} \rangle = \mathrm{Power}(\{A, B, C\})$. (In 2025, we did this example on Jan 22.) For the $\mathcal{C}_i(\mathsf{K}_{\mathsf{abs}})$ we introduce the following bases, $\mathcal{B}_i$:

(1) for $\mathcal{C}_2$: $\mathcal{B}_2 = \{[A, B, C]\}$ (so $\dim(\mathcal{C}_2) = 1$);
(2) for $\mathcal{C}_1$: $\mathcal{B}_1 = \{[A, B], [A, C], [B, C]\}$ (so $\dim(\mathcal{C}_1) = 3$); and
(3) for $\mathcal{C}_0$: $\mathcal{B}_0 = \{[A], [B], [C]\}$ (so $\dim(\mathcal{C}_0) = 3$).

With respect to these bases we have

$$\partial_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}_{\mathcal{B}_2, \mathcal{B}_1} \qquad \text{since } \partial_2[A, B, C] = [B, C] - [A, C] + [A, B],$$

$$\partial_1 = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}_{\mathcal{B}_1, \mathcal{B}_0}, \qquad \text{(compare with Example 4.4).}$$

A tedious (but short) computation shows that $Z_1 \stackrel{\text{def}}{=} \ker(\partial_1)$ equals $B_1 \stackrel{\text{def}}{=} \mathrm{Image}(\partial_2)$ (indeed, both are one-dimensional); we easily see that $Z_2 = \ker \partial_2 = 0$ (and, by definition $B_2 = 0$); and, similar to in Example 4.4, $Z_0 = \mathcal{C}_0(\mathsf{K}_{\mathsf{abs}})$ and [equation below corrected March 17, 2025 by a student]

$$B_0 \stackrel{\text{def}}{=} \mathrm{Image}(\partial_1) = \{\beta_1[A, B] + \beta_2[A, C] + \beta_3[B, C] \mid \beta_1 + \beta_2 + \beta_3 = 0\}$$

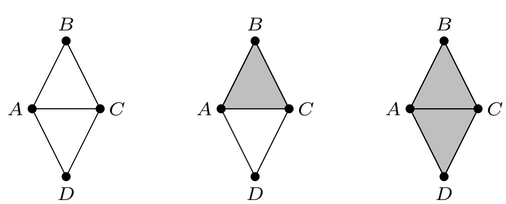
(which is two-dimensional). Hence

$$H_2^{\mathrm{simp}}(\mathsf{K}_{\mathsf{abs}}) = 0, \quad H_1^{\mathrm{simp}}(\mathsf{K}_{\mathsf{abs}}) = 0, \quad H_0^{\mathrm{simp}}(\mathsf{K}_{\mathsf{abs}}) \simeq \mathbb{R}.$$

Example 4.9. Let $S_1 = \{A, B, C\}$ and $S_2 = \{A, C, D\}$, and

$$\mathsf{K}^2_{\mathsf{abs}} = \langle S_1, S_2 \rangle, \quad \mathsf{K}^1_{\mathsf{abs}} = \mathsf{K}^2_{\mathsf{abs}} \setminus \{S_2\}, \quad \mathsf{K}^0_{\mathsf{abs}} = \mathsf{K}^1_{\mathsf{abs}} \setminus \{S_1\}.$$

In other words, $\mathsf{K}^0_{\mathsf{abs}}$ is a graph, namely the complete graph on vertex set A, B, C, D with the edge $\{B, D\}$ omitted; $\mathsf{K}^1_{\mathsf{abs}}$ has the 2-face S_1 added, and $\mathsf{K}^2_{\mathsf{abs}}$ has S_1 and S_2 added.



$\mathsf{K}^0_{\mathsf{abs}}$ $\qquad$ $\mathsf{K}^1_{\mathsf{abs}}$ $\qquad$ $\mathsf{K}^2_{\mathsf{abs}}$

In Exercise A.6 we compute the Betti numbers of $\mathsf{K}^0_{\mathsf{abs}}, \mathsf{K}^1_{\mathsf{abs}}, \mathsf{K}^2_{\mathsf{abs}}$. This exercise shows that for $i = 1, 2, 3$ we have $H_0(\mathsf{K}^i_{\mathsf{abs}}) \simeq \mathbb{R}$ and $H_1(\mathsf{K}^i_{\mathsf{abs}}) \simeq \mathbb{R}^{3-i}$, where one possible basis for $H_1(\mathsf{K}^0_{\mathsf{abs}})$ is $\tau_1 = [A, B] + [B, C] + [C, A]$ and $\tau_2 = [A, C] + [C, D] + [D, A]$. In $\mathsf{K}^2_{\mathsf{abs}}$, we have $\partial_2[A, B, C] = \tau_1$ and $\partial_2[A, C, D] = \tau_2$.

Remark 4.10. Once we know the homology groups for a *cone* (see below), and we realize that $\mathsf{K}^2_{\mathsf{abs}}$ is an example of a *cone*, then we will know that $\beta_0(\mathsf{K}^2_{\mathsf{abs}}) = 1$ and $\beta_1(\mathsf{K}^2_{\mathsf{abs}}) = \beta_2(\mathsf{K}^2_{\mathsf{abs}}) = 0$. This easily allows to compute $\beta_j(\mathsf{K}^i_{\mathsf{abs}})$ for all j and for $i = 0, 1$ as well.

4.5. **General Simplicial Homology and Cones.** For an arbitrary abstract simplicial, $\mathsf{K}_{\mathsf{abs}}$, we similarly define the simplicial homology groups $H_i^{\mathrm{simp}}(\mathsf{K}_{\mathsf{abs}})$. Namely, for any k-face $\{u_0, \ldots, u_k\}$ in $\mathsf{K}_{\mathsf{abs}}$ we introduce the symbol $[u_0, \ldots, u_k]$, with the understanding that the same symbol with $u_0, \ldots, u_k$ permuted represents the same thing times ± 1, depending on the sign of the permutation; i.e., for any permutation σ of $\{0, 1, \ldots, k\}$,

$$[u_{\sigma(0)}, \ldots, u_{\sigma(k)}] = \mathrm{sign}(\sigma)[u_0, \ldots, u_k].$$

We use $\mathcal{C}_k = \mathcal{C}_k(\mathsf{K}_{\mathsf{abs}})$ to denote the set of *k-form of $\mathsf{K}_{\mathsf{abs}}$*, which are any formal linear combination of above symbols. We define $\partial_k \colon \mathcal{C}_k \to \mathcal{C}_{k-1}$ as the unique linear transformation with

$$\partial_k[u_0, \ldots, u_k] = \sum_{i=0}^{k} [u_0, \ldots, \widehat{u_i}, \ldots, u_k](-1)^i.$$

We easily verify that $\partial_{k-1}\partial_k = 0$ for all k, and we define the k-th homology group of $\mathsf{K}_{\mathsf{abs}}$ to be

$$H_k^{\mathrm{simp}}(\mathsf{K}_{\mathsf{abs}}) = \ker(\partial_k)/\mathrm{Image}(\partial_{k+1}).$$

We will spend a lot of time studying $H_k^{\mathrm{simp}}(\mathsf{K}_{\mathsf{abs}})$ and the analogous homology groups of *topological spaces*. We will also provide many tools to compute theses groups. Here is a example of an abstract simplicial complex where the simplicial homology groups are well-known.

Definition 4.11. If $\mathsf{K}_{\mathsf{abs}}$ is a simplicial complex, P is not a vertex of $\mathsf{K}_{\mathsf{abs}}$, then $\mathrm{Cone}_P(\mathsf{K}_{\mathsf{abs}})$ is the simplicial complex

$$\mathrm{Cone}_P(\mathsf{K}_{\mathsf{abs}}) = \bigcup_{A \in \mathsf{K}_{\mathsf{abs}}} \{A, A \cup P\}.$$

Conversely, if $\mathsf{L}_{\mathsf{abs}}$ is a simplicial complex, and P is a vertex of $\mathsf{L}_{\mathsf{abs}}$ then we say that *$\mathsf{L}_{\mathsf{abs}}$ simplicially contracts at P* if for all $A \in \mathsf{L}_{\mathsf{abs}}$ we have $A \cup \{P\} \in \mathsf{L}_{\mathsf{abs}}$.

Then we easily see that $\mathsf{L}_{\mathsf{abs}}$ simiplicially contracts at P iff $\mathsf{L}_{\mathsf{abs}}$ can be written as $\mathrm{Cone}_P(\mathsf{K}_{\mathsf{abs}})$ for some $\mathsf{K}_{\mathsf{abs}}$.

Example 4.12. For any distinct $u_0, \ldots, u_i$, let $U = \{u_0, \ldots, u_i\}$ and $U' = \{u_0, \ldots, u_{i-1}\}$. Then

$$\langle U \rangle = \mathrm{Cone}_{u_i} \langle U' \rangle.$$

We will soon show that for any $\mathsf{L}_{\mathsf{abs}} = \mathrm{Cone}_P(\mathsf{K}_{\mathsf{abs}})$ we have $H_0(\mathsf{L}_{\mathsf{abs}}) \simeq \mathbb{R}$ (since $\mathsf{L}_{\mathsf{abs}}$ is *connected*) and $H_i(\mathsf{L}_{\mathsf{abs}}) = 0$ for $i \geq 1$.

Remark 4.13. In class on Jan 22 and Jan 24, 2025, we decided to prove the above result, which is essentially the first proof of Theorem 6.1. Let us review the way we stated the proof in class:

(1) Let $\mathcal{P}_i \in \mathcal{C}_i(\mathsf{L}_{\mathsf{abs}})$ be those linear combinations of the form $[u_0, \ldots, u_i]$ where $u_0 = P$. If $\tau \in Z_i(\mathsf{L}_{\mathsf{abs}}) \ker(\partial_i)$, then we claim we can find a τ' such that

$\tau' \equiv \tau \pmod{B_{i+1}}$ and $\tau' \in \mathcal{P}_i$. To do so, notice that if $u_0, \ldots, u_i$ are distinct vertices of $\mathsf{L}_{\mathsf{abs}}$, and P is distinct from $u_0, \ldots, u_i$, then

$$\begin{aligned}\partial_{i+1}[P, u_0, \ldots, u_i] &= [u_0, \ldots, u_i] - [P, u_1, \ldots, u_i] + [P, u_0, u_2, \ldots, u_i] - \cdots, \\ &= [u_0, \ldots, u_i] + \rho\end{aligned}$$

where $\rho \in \mathcal{P}_i$. Hence

$$[u_0, \ldots, u_i] \equiv [P, u_1, \ldots, u_i] - [P, u_0, u_2, \ldots, u_i] + \cdots \pmod{B_{i+1}}.$$

Doing this to each such term $[u_0, \ldots, u_i]$ gives us a $\tau' \in \mathcal{P}_i$ with $\tau' \equiv \tau \pmod{B_{i+1}}$.

(2) Since τ' and τ differ from an element in the image of ∂_{i+1}, and since $\partial_i \partial_{i+1} = 0$, we have $\partial_i \tau' = \partial_i \tau$; since $\partial_i \tau = 0$, we have $\partial_i \tau' = 0$.

(3) We claim that if $\tau' \in \mathcal{P}_i$ and $\partial_i \tau' = 0$, then $\tau' = 0$. This is because τ' is a linear combination of terms of the form $[P, u_1, \ldots, u_i]$ in $\mathcal{C}_i$, i.e., where $[u_1, \ldots, u_i] \in \mathcal{C}_{i-1}(\mathsf{K}_{\mathsf{abs}})$. So we may write

$$\tau' = \sum_{[u_i, \ldots, u_i]} \alpha_{u_1, \ldots, u_i} [P, u_1, \ldots, u_i]$$

where each set of size $i-1$, $\{u_1, \ldots, u_i\} \in \mathsf{K}_{\mathsf{abs}}$ appears once above (in some "orietnation" $[u_1, \ldots, u_i]$); since

$$\partial_i [P, u_1, \ldots, u_i] \equiv [u_1, \ldots, u_i] \pmod{P_{i-1}},$$

(i.e., $[u_1, \ldots, u_i]$ is the only term where a "P" does not occur), we have

$$0 = \partial_i \tau' \equiv \sum_{[u_i, \ldots, u_i]} \alpha_{u_1, \ldots, u_i} [u_1, \ldots, u_i] \pmod{P_{i-1}},$$

and hence $\alpha_{u_1, \ldots, u_i} = 0$ for all terms in this sum.

(4) Since $\tau' = 0$, and $\tau' \equiv \tau \pmod{)} B_i$, we have $\tau \in B_i$ and therefore τ is 0 in $H_i(\mathsf{L}_{\mathsf{abs}}) = Z_i(\mathsf{L}_{\mathsf{abs}})/B_i(\mathsf{L}_{\mathsf{abs}})$.

However, we will likely save the second proof for a bit later, when we are more comfortable with chains.

Example 4.14. In class in 2025, we gave the following example: say that $\mathsf{K}_{\mathsf{abs}} = \langle \{A\}, \{B, C\}, \{D\} \rangle$, and $\mathsf{L}_{\mathsf{abs}} = \mathrm{Cone}_P(\mathsf{K}_{\mathsf{abs}})$. Then $\tau = [B, P] + [P, C] + [C, B]$ is a cycle. So we "clear out" the $[C, B]$ term: since

$$\partial_2([P, C, B]) = [C, B] - [P, B] + [P, C],$$

we have

$$[C, B] \equiv [P, B] - [P, C] \pmod{B_1(\mathsf{L}_{\mathsf{abs}})}.$$

Hence we have $\tau' \equiv \tau \pmod{B_1(\mathsf{L}_{\mathsf{abs}})}$, where

$$\tau = [B, P] + [P, C] + [C, B], \quad \tau' = [B, P] + [P, C] + \big([P, B] - [P, C]\big)$$

so τ' is already $= 0$ (!). Hence $\tau \equiv 0 \pmod{B_1(\mathsf{L}_{\mathsf{abs}})}$, so τ is in $B_1(\mathsf{L}_{\mathsf{abs}})$ and so τ is 0 in the quotient $H_1(\mathsf{L}_{\mathsf{abs}}) = Z_1(\mathsf{L}_{\mathsf{abs}})/B_1(\mathsf{L}_{\mathsf{abs}})$

Remark 4.15. We will be interested in cones for two reasons: First, the *suspension* is an important construction built out of two cones. Second, if $\mathsf{K}_{\mathsf{abs}}$ is a simplicial complex and P a vertex, the set

$$\mathrm{LargestContractible}_P(\mathsf{K}_{\mathsf{abs}}) = \{A \in \mathsf{K}_{\mathsf{abs}} \mid A \cup P \in \mathsf{K}_{\mathsf{abs}}\} \tag{14}$$

is the largest subcomplex of $\mathsf{K}_{\mathsf{abs}}$ that simplicially contracts at P; it is of importance for many reasons, including: (1) a way to simplify $\mathsf{K}_{\mathsf{abs}}$; (2) a way to find a *good cover* of A; (3) a way to show the equality of two *triangulations* of the same *topological space*, (4) etc. The subcomplex (14) is sometimes called *the star of K_{abs} at P*, although more often the *star* refers to only those sets in (14) that do not contain P.

4.6. **Some More Examples.** It is not a bad idea to compute the groups $H_i^{\mathrm{simp}}(\mathsf{K}_{\mathsf{abs}})$ for some examples beyond graphs, although this can wait. Munkres textbook [Mun84] has a bunch of examples, including some 2-dimensional examples, based on surfaces, Sections 1.5 and 1.6 (and beyond).

4.7. The Zero-th Homology Group and Betti Number.

4.7.1. *Connected Components of Graphs.*

Definition 4.16. Let $G = (V, E)$ be a simple graph. A *walk in G* refers to a sequence

$$(v_0, \ldots, v_k) \quad \text{such that } \{v_i, v_{i+1}\} \in E \text{ for } 0 \le i \le k-1;$$

we call this a walk of *length k*, *from v_0* (or *beginning in v_0*), *to v_k* (or *ending in v_k*). We say that $v, v' \in V$ are *connected* (in G), writing $v \sim v'$, if there is a walk from v to v'.

We easily see that V is partitioned into subsets $V_1, \ldots, V_r$ where each V_i is a maximal subset of vertices that are connected to one another; said otherwise, $v \sim v'$ is an *equivalence relation*, and hence its *equivalence classes* partitions V into subsets $V_1, \ldots, V_r$ of equivalent vertices.

Definition 4.17. Let $G = (V, E)$ be a simple graph. A *connected component of vertices of G* refers to a maximal subset $V' \subset V$ of connected vertices of G. If $E' \subset E$ is the (possibly empty) subset of edges between elements of V', we refer to $G' = (V', E')$ as a *connected component subgraph of G*. We often simply write *connected component* the distinction is understood (or unimportant, i.e., when speaking only of vertices). We say that G is *connected* if it nonempty and has a single connected component.[3]

It follows that $G = (V, E)$ is the "disjoint union" of its connected components $G_i = (V_i, E_i)$.

4.7.2. *Walks and $B_0 = \mathrm{Image}(\partial_1)$.* If $(v_0, \ldots, v_k)$ is any walk in a graph, $G = (V, E)$, then

$$\tau = [v_0, v_1] + [v_1, v_2] + \cdots + [v_{k-1}, v_k] \in \mathcal{C}_1(G),$$

and

$$\partial_1(\tau) = \big([v_1] - [v_0]\big) + \cdots + \big([v_k] - [v_{k-1}]\big) = [v_k] - [v_0]. \tag{15}$$

Notice that ∂_1 has its image spanned by elements of $\mathcal{C}_0(G)$ of the form $[v'] - [v]$, and these are of the form

$$\sum_{v \in V} \alpha_v v \quad \text{such that } \textstyle\sum_v \alpha_v = 0.$$

[3] Here we don't care if we "connected components" refers to a subset of vertices or the subgraphs.

It follows that if $\sum_v \alpha_v v \in \mathcal{C}_0(G)$ and is in the image of ∂_1, then $\sum_v \alpha_v = 0$. It follows that $[v] \in \mathcal{C}_0(G)$ is not in the image of ∂_1. It will easily follow that $H_0^{\mathrm{simp}}(G)$ is one-dimensional if G is connected; however, it will be convenient introduce some standard notation and terminology regarding quotient spaces.

4.7.3. *Quotient Spaces.* Here we review some essentially standard notation for quotients of vector spaces. For more review, we refer to the reader to a sequence of exercises beginning with Exercise C.2.

Notation 4.18 (Quotient of two vector spaces)**.** Let $U \subset W$ be two vector spaces over a field, $\mathbb{F}$. For $w \in W$, we use the notation $w + U$ to refer to

$$w + U = \{w + u \ |u \in U\}.$$

We refer to any set $w + U$ as a *U-coset of W*. [The sets of the form $w + U$ form a vector space in an evident fashion, denoted W/U, and called the *quotient of W by U* or *W modulo U*.] We will also use *w in W/U* to refer to $w + U$ when confusion is unlikely. We also write $w_1 \equiv_U w_2$ when $w_1, w_2 \in W$ are in the same element of W/U. For $w \in W$, we will often say *the image of w in W/U* or simply *w in W/U* to refer to $w + U$. The map $w \mapsto w + U$ is called the *natural map* $W \to W/U$.

A standard argument shows that $\dim(W/U) = \dim(W) - \dim(U)$.

Example 4.19. Let $W = \mathbb{R}^3$, and $U \subset W$ be given by

$$U = \{\mathbf{u} = (u_1, u_2, u_3) \mid u_1 + u_2 + u_3 = 0\}.$$

Then $\dim(U) = 2$, and W/U refers to the cosets $w+U$ with $w \in \mathbb{R}^3$; $\dim(W/U) = 1$. Notice that $U^\perp \subset W$ is generated by $(1, 1, 1)$, but one should be careful not to identify $U^\perp$ with W/U; W/U is a space of U-cosets, and there is no "canonical" basis vector for the 1-dimensional space W/U. Notice also that

$$(1, 0, 0)+W = (0, 1, 0)+W = (0, 0, 1)+W = -\bigl((-1, 0, 0)+W\bigr) = 2\bigl((1/4, 1/4, 0)+W\bigr)$$

We refer the reader to Exercises C.2 for more examples.

Notice than in defining

$$H_i(\mathcal{C}) = Z_i(\mathcal{C})/B_i(\mathcal{C}) = \ker(\partial_i)/\mathrm{Image}(\partial_{i+1}),$$

we have already implicitly used quotient spaces.

Hence if v, v' are connected vertices in a graph $G = (V, E)$, one can rewrite (15) in any of the equivalent ways: (1) $v \equiv_{\mathrm{Image}(\partial_1)} v'$, (2) $v \equiv_{B_0} v'$, (3) $v \in v' + B_0$, (4) $v + B_0 = v' + B_0$, (5) etc.

Notice that the quotient is also defined for any two *groups*[4] $U \subset W$. For example, when we write $\mathbb{Z}/3\mathbb{Z}$, we take $U = 3\mathbb{Z} = \{\dots, -3, 0, 3, \dots\}$ and $W = \mathbb{Z}$, and so $\mathbb{Z}/3\mathbb{Z}$ has three elements:

(1) $0 + U = U = 3\mathbb{Z}$,
(2) $1 + U = 1 + 3\mathbb{Z} = \{\dots, -5, -2, 1, 4, 7, \dots\}$, and
(3) $2 + U = 2 + 3\mathbb{Z} = \{\dots, -4, -1, 2, 5, 8, \dots\}$.

[4]A *group* is a set with a $+$ operation that is associative, and has a 0 and inverses. However, most authors use $+$ to imply that the group is commutative, and would otherwise write $g_1 \cdot g_2$ or $g_1 g_2$ to denote the group operation applied to g_1, g_2 in the group.

This is equivalent to speaking of "the integers modulo 3," which in computer science we understand to mean the set $\{0,1,2\}$ and addition is $a+b \stackrel{\text{def}}{=} a+b \bmod 3$.

In particular, if $U \subset W$ are vector spaces (over any field), then the are both (commutative) groups under the $+$ operation, and W/U becomes a vector space (over the same field).

4.7.4. *$H_0(G)$ if G is connected.*

Proposition 4.20. *Let $G=(V,E)$ be a connected, nonempty graph, and $v_1 \in V$. Then $H_0^{\text{simp}}(G)$ is isomorphic to $\mathbb{R}$ and is spanned by the image of $[v_1]$ in $H_0^{\text{simp}}(G)=\mathcal{C}_0(G)/B_0$.*

Proof. If $\sum_v \alpha_v v$ is in the image of ∂_1, then $\sum_v \alpha_v = 0$. Hence $[v_1] \in \mathcal{C}_0(G)$ is not in $B_0 = \operatorname{Image}(\partial_1)$, and so $[v_1]+B_0 = [v_1]+\operatorname{Image}(\partial_1)$ is nonzero in $H_0^{\text{simp}}(G)$. If $\sum_v \alpha_v \in \mathcal{C}_0(G)$ is an arbitrary element, then there is a walk from v to v_1 in G, and hence

$$\sum_v \alpha_v([v]-[v_1]) \in B_0 = \operatorname{Image}(\partial_1),$$

and it follows that

$$\sum_v \alpha_v[v] \equiv_{B_0} \left(\sum_v \alpha_v\right)[v_1].$$

Hence any element of $H_0^{\text{simp}}(G)=\mathcal{C}_0(G)/B_0$ can be written as a scalar multiple of $[v_1]$ in $H_0^{\text{simp}}(G)$. $\square$

At this point we will often write B_0 instead of $\operatorname{Image}(\partial_1)$.

Another way to describe Proposition 4.20 is that using $\mathbb{R}^V$ to denote the functions $V \to \mathbb{R}$, and $\mathbf{1}=(1,1,\ldots,1) \in \mathbb{R}^V$, the above proof shows that

$$B_0 = \operatorname{Image}(\partial_1) = \mathbf{1}^{\text{perp}} = \{\boldsymbol{\alpha} \in \mathbb{R}^V \mid \boldsymbol{\alpha}\cdot\mathbf{1} = 0\},$$

where $\cdot$ denotes the dot product, i.e.,

$$\boldsymbol{\alpha}\cdot\mathbf{1} = \sum_{i=1}^n \alpha_i = \alpha_1+\cdots+\alpha_n,$$

where $|V|=n$ and we identify $\mathbb{R}^V$ with $\mathbb{R}^n$. Since

$$H_0(\mathcal{C}) \stackrel{\text{def}}{=} \mathcal{C}_0/B_0 = \mathbb{R}^V/\mathbf{1}^{\text{perp}},$$

it follows that $H_0(\mathcal{C})$.

Remark 4.21. For any subspace $U \subset \mathbb{R}^n$ we have that each vector in $\mathbb{R}^n$ can be written uniquely as a vector in U plus one in $U^\perp$, where

$$U^\perp \stackrel{\text{def}}{=} \{\mathbf{w} \in \mathbb{R}^n \mid \forall \mathbf{u} \in U,\ \mathbf{w}\cdot\mathbf{u} = 0\}.$$

From this it follows that $\mathbb{R}^n/U$ is *isomorphic* to $U^\perp$. However, $\mathbb{R}^n/U$ and $U^\perp$ are not the same space. When we replace homology over $\mathbb{R}$ with homology over a general field or ring, we will no longer have that $\mathbb{R}^n/U$ is isomorphic to what one should call $U^\perp$; indeed, U and $U^\perp$ can intersect.

Example 4.22. Do the example of $\mathsf{K}_{\mathsf{abs}} = \operatorname{Power}(\{A,B,C\}) \setminus \{A,B,C\}$, the complete graph on 3 vertices. Do another example: a complete graph, a tree, etc.

4.7.5. *$H_0(\mathsf{K}_{\mathsf{abs}})$ for connected abstract simplicial complexes.* Note that if $\mathsf{K}_{\mathsf{abs}}$ is an arbitrary abstract simplicial complex, since

$$H_0(\mathsf{K}_{\mathsf{abs}}) \stackrel{\mathrm{def}}{=} Z_0/B_0 = \ker(\partial_0)/\mathrm{Image}(\partial_1), \tag{16}$$

it follows that $H_0(\mathsf{K}_{\mathsf{abs}}) = H_0(G)$ (not just isomorphic to, but actual equality!) when $G = (V, E)$ is the graph associated to $\mathsf{K}_{\mathsf{abs}}$ (based on the 0- and 1-simplexes of $\mathsf{K}_{\mathsf{abs}}$).

For this reason, we make the following definitions.

Definition 4.23. If $\mathsf{K}_{\mathsf{abs}}$ is a general simplicial complex with vertex and edge sets V, E, we define a *connected component of vertices of $\mathsf{K}_{\mathsf{abs}}$* to be that of the graph $G = (V, E)$; for any such connected component $V' \subset V$, we similarly define the *connected component subcomplex $\mathsf{K}_{\mathsf{abs}}$ with vertex set V'* to be the subcomplex $\mathrm{Power}(V') \cap \mathsf{K}_{\mathsf{abs}}$.

The following proposition is implied by (16).

Proposition 4.24. *Let $\mathsf{K}_{\mathsf{abs}}$ be an abstract simplicial complex whose underlying graph is $G = (V, E)$ (i.e., V, E are the vertex set and edge set, respectively, of $\mathsf{K}_{\mathsf{abs}}$). Then $H_0(\mathsf{K}_{\mathsf{abs}}) = H_0(G)$, and $\beta_0(\mathsf{K}_{\mathsf{abs}})$ is the number of connected components of G.*

4.7.6. *Graphs that are not connected: first proof.*

Proposition 4.25. *Let G be a connected, nonempty graph. Let $V_1, \ldots, V_r$ be the vertices of the connected components of G. For $i \in [r]$, let $v_i \in V_i$. Then $H_0^{\mathrm{simp}}(G)$ is isomorphic to $\mathbb{R}$ and is spanned by the images of $[v_1], \ldots, [v_r]$ in the quotient space $H_0^{\mathrm{simp}}(G) = \mathcal{C}_0(G)/\mathrm{Image}(\partial_1)[v_0]$.*

Proof. The proof of the case when G is connected shows that for any i, $\tau = \sum_{v \in V_i} \alpha_v[v]$ is equal to $(\sum_v \alpha_v)[v_i]$ modulo $B_0 = \mathrm{Image}(\partial_1)$. Hence $H_0^{\mathrm{simp}}(G) = \mathcal{C}_0(G)/B_0$ is generated by the vectors $[v_1], \ldots, [v_r]$ in $H_0^{\mathrm{simp}}(G)$. Now we claim that $[v_1], \ldots, [v_r]$ are linearly independent in $H_0^{\mathrm{simp}}(G) = \mathcal{C}_0(G)/B_0$: if not, then for some $[v_i]$ can be written as a linear combination of the other $[v_1], \ldots, [v_r]$; we may assume $i = 1$, and hence

$$[v_1] = \sum_{i=2}^{k} \beta_i [v_i] \quad \text{(in their image in } \mathcal{C}_0(G)/B_0(G),$$

i.e., in $\mathcal{C}_0(G)$ we have

$$[v_1] = \sum_{i=2}^{k} \beta_i [v_i] + \partial_1 \tau,$$

where $\tau \in \mathcal{C}_1(G)$, and hence

$$\tau = \sum_{\{v, v'\} \in E} \alpha_{v,v'} [v, v'].$$

Let $E_i \subset E$ be the set of edges between elements of V_i. Since each v, v' with $\{v, v'\} \in E$ lies in exactly one of E_i, we can write

$$\tau = \tau_1 + \cdots + \tau_r,$$

where τ_i contains only those 1-forms that are linear combinations of $[v, v']$ with $\{v, v'\} \in E_i$. For $V' \subset V$, let $\mathbb{1}_{V'} \in \mathbb{R}^V$ be the vector whose V' components are 1, and whose other components are 0. Viewing $[v_i]$ as a function in $\mathbb{R}^V$ (which is the same thing as a function $V \to \mathbb{R}$), taking the dot product of $\mathbb{1}_{V_1}$ with the equation

$$[v_1] = \sum_{i=2}^{r} \beta_i [v_i] + \sum_{i=1}^{r} \partial_1(\tau_i)$$

(which is the same thing as viewing both sides as functions $V \to \mathbb{R}$ and summing over all the values of this function on $v \in V_1$) we have all the dot products (or sums over V') of the RHS are 0, and the dot product on the LHS is 1. $\square$

Notice that this proof really works because G is a disjoint union of $G_i = (V_i, E_i)$. A better conceptual way to understand what's going on is that there is a "direct sum decomposition"

$$\mathcal{C}_1(G) \simeq \mathcal{C}_1(G_1) \oplus \cdots \oplus \mathcal{C}_1(G_r),$$

and similarly for $\mathcal{C}_0$, and the map ∂_1 "factors through" these two direct sum decompositions. Another way to say this is that ∂_1 is a "block diagonal matrix" with respect to bases for $\mathcal{C}_1(G)$ and $\mathcal{C}_0(G)$ built from individual bases for $\mathcal{C}_1(G_i)$ and $\mathcal{C}_0(G_i)$ for all i. In this way it turns out that

$$H_0^{\mathrm{simp}}(G) \simeq H_0^{\mathrm{simp}}(G_1) \oplus \cdots \oplus H_0^{\mathrm{simp}}(G_r)$$

(and similarly for all H_j^{simp}, and for all simplicial complexes $\mathsf{K}_{\mathsf{abs}}$ that are not connected). Let us formalize this.

4.7.7. *Direct sums, for graphs that are not connected.* We will need to review block matrices and direct sums at this point.

If $U_1, \ldots, U_r$ are $\mathbb{R}$-vector spaces, the *direct sum of* $U_1, \ldots, U_r$, denoted $U_1 \oplus \cdots \oplus U_r$, is the vector space

$$U_1 \oplus \cdots \oplus U_r = \{(u_1, \ldots, u_r) \mid u_i \in U_i\},$$

endowed with its evident structure as a $\mathbb{R}$-vector space (i.e., addition and scalar multiplication are performed "component-wise"[5]). If $U_1, \ldots, U_r \subset U$ are subspaces, we say that U *decomposes as a direct sum of subspaces* $U_1, \ldots, U_n$ or U *is an internal direct sum of* $U_1, \ldots, U_n$ if any (all) of the equivalent conditions hold:

1. each vector in U can be written uniquely as a linear combination of vectors in the U_i;
2. the map $f\colon (U_1 \oplus \ldots \oplus U_n) \to U$ taking $(u_1, \ldots, u_n)$ to $u_1 + \cdots + u_n$ is an isomorphism;
3. $\mathrm{Span}(U_1, \ldots, U_r)$ (the span of $U_1, \ldots, U_r$, i.e., the smallest subspace containing all these vectors) equals U, and $\dim(U_1) + \ldots + \dim(U_r) = \dim(U)$;
4. the subspaces $U_1, \ldots, U_r$ are linearly independent (i.e., $u_i \in U_i$ and $u_1 + \cdots + u_r = 0$ implies that $u_i = 0$) and $\dim(U_1) + \ldots + \dim(U_r) = \dim(U)$;
5. etc.

If so, and if W decomposes as a direct sum $W_1, \ldots, W_m$, then any linear map $\mathcal{L}\colon U \to W$ decomposes into "blocks" $\mathcal{L}_{ij}\colon U_j \to W_i$, where for $u_j \in U_j$, $\mathcal{L}(u_j) = \mathcal{L}_{1,j}(u_j) + \cdots + \mathcal{L}_{m,j}(u_r)$. By choosing bases for the U_j with $j \in [r]$ and W_i with $i \in [r']$, $\mathcal{L}$ becomes a "block matrix" whose "j-th block columns" represent the image of $\mathcal{L}$ restricted to U_j, as it decomposes uniquely as a sum of $W_1, \ldots, W_m$. If, in addition, $r = r'$ and $\mathcal{L}(u_i) \in W_i$ for all $u_i \in U_i$, or equivalently $\mathcal{L}_{ji} = 0$ for $j \neq i$,

[5] For example, $(u_1, \ldots, u_r) + (u_1', \ldots, u_r') = (u_1 + u_1', \ldots, u_r + u_r')$.

we say that $\mathcal{L}$ *factors through* these direct sum decompositions. In this case $\mathcal{L}$ is a "block diagonal matrix."

We will give some exercises to make these ideas concrete, starting with Exercise C.3.

4.7.8. *The Case of General Graphs.* Now we address the case where $G = (V, E)$ is not connected. In this case, if $V_1, \ldots, V_r$ are the vertex connected components of G, then G is the disjoint union of (V_i, E_i); note that ∂_1 *factors* through the *decompositions*

$$\mathcal{C}_1(G) = \mathcal{C}_1(G_1) \oplus \cdots \oplus \mathcal{C}_1(G_r), \quad \mathcal{C}_0(G) = \mathcal{C}_0(G_1) \oplus \cdots \oplus \mathcal{C}_0(G_r),$$

i.e., ∂_1 takes $\mathcal{C}_1(G_i)$ to $\mathcal{C}_0(G_i)$, i.e., ∂_1 is an $r \times r$ *diagonal block matrix* with respect to a choice of basis vectors for $\mathcal{C}_1(G_i)$ and $\mathcal{C}_0(G_i)$. From this it follows that

$$H_0^{\mathrm{simp}}(\mathcal{C}) = \bigoplus_{i=1}^{r} H_0^{\mathrm{simp}}(\mathcal{C}).$$

This proves the main result of this subsection.

Proposition 4.26. *Let G be a connected, nonempty graph. Let $v_1, \ldots, v_r$ be a choice of one vertex in each connected component of G. Then $H_0^{\mathrm{simp}}(G)$ is isomorphic to $\mathbb{R}$ and is spanned by the images of $[v_1], \ldots, [v_r]$ in the quotient space $H_0^{\mathrm{simp}}(G) = \mathcal{C}_0(G)/\mathrm{Image}(\partial_1)[v_0]$.*

4.8. $Z_1(G)$ **really represents "cycles".** In Exercise A.7 we show that in a graph G, an element of $Z_1(G) = \ker(\partial_1)$ arises from "cycles" in G. This justifies the "Z" in Z_1, which comes from the German term *Zyklus*; this idea is helpful when considering what Z_i means for all $i \geq 1$.

4.9. **Incidence Matrix and Laplacians.** [This subsection may be skipped or covered later.]

If $G = (V, E)$ is a graph, then $\partial_1 \colon \mathcal{C}_1(G) \to \mathcal{C}_0(G)$ is nothing other than the classical "incidence matrix:" to build the incidence matrix, we fix an *orientation* for each edge, meaning that for each $e = \{v, v'\} \in E$ we pick some order for v, v' (either (v, v') or (v', v)); the order doesn't matter. Once we've done so then we can identify $\mathcal{C}_1(G)$ with $\mathbb{R}^E$, and $\partial_1 \mathbb{R}^E \to \mathbb{R}^V$ is a $|V| \times |E|$ matrix where the column corresponding to an oriented edge (v, v') has a $+1$ in the row corresponding to v', a -1 in that corresponding to v, and 0's elsewhere. We then have that

$$\partial_1 \partial_1^{\mathrm{T}} \colon \mathbb{R}^V \to \mathbb{R}^V$$

is a positive semidefinite matrix which we call the *graph Laplacian* of G, denoted $\Delta = \Delta_G = \partial_1 \partial_1^{\mathrm{T}}$. For any graph we define:

(1) the *adjacency matrix* of G, $A = A_G$, to be the $|V| \times |V|$, $0, 1$ matrix with a 1 in entry (v, v') iff $\{v, v'\} \in E$; and
(2) the *degree counting matrix* of G, $D = D_G$, is the diagonal $|V| \times |V|$ matrix whose (v, v)-entry is the *degree of v in G*, i.e., the number of edges that v is incident upon (this also equals the row sum of A corresponding to v).

We easily see that $\Delta_G = D_G - A_G$.

The matrices, $\Delta_G, A_G, \partial_1(G)$, are the foundational matrices in the rich field of *algebraic graph theory*.

There is also an *edge Laplacian*, $\partial_1^{\mathrm{T}}\partial_1$, mapping $\mathbb{R}^E \to \mathbb{R}^E$, although it is less prominent in classical algebraic graph theory. We use Δ_G^{edge} to denote the edge Laplacian of G, and use Δ_G^{vert} to denote the classical graph Laplacian $\Delta_G = D_G - A_G$ described above.

Definition 4.27. By a *harmonic* 0*-form* (respectively, *harmonic* 1*-form*) in a graph, we mean an element of the kernel of Δ_G^{vert} (respectively, Δ_G^{edge}. For $i = 0, 1$, we use $\mathcal{H}_i(G)$ to denote the space of harmonic i-forms of G.

Theorem 4.28. *Let $G = (V, E)$ be a simple graph. Then if $\tau \in \mathcal{H}_i(G)$ we have $\partial_i \tau = 0$. Moreover, there is an isomorphism $\mathcal{H}_i(G) \to H_i^{\mathrm{simp}}(G)$ taking $\tau \in \mathcal{H}_i(G)$ to its class in $H_i^{\mathrm{simp}}(G) = \ker(\partial_i)/\mathit{Image}(\partial_{i+1})$.*

We easily see that $\mathcal{H}_0(G)$ is the vector space of functions $V \to \mathbb{R}$ that are constant on each connected component (of vertices) of G.

The above will be generalized to abstract simplicial complexes of arbitrary dimension.

We remark that the eigenfunctions of Δ_G and A_G can give useful information about the vertices of G. This is a big field; having some intuition about this can help to understand Laplacians in abstract simplicial complexes of higher dimension.

4.10. **Laplacians in Simplicial Homology.**

For an arbitrary abstract simplicial, $\mathsf{K}_{\mathsf{abs}}$, we similarly define the simplicial homology groups $H_i^{\mathrm{simp}}(\mathsf{K}_{\mathsf{abs}})$. Namely, for any k-face $\{u_0, \ldots, u_k\}$ in $\mathsf{K}_{\mathsf{abs}}$ we introduce the symbol $[u_0, \ldots, u_k]$, with the understanding that the same symbol with $u_0, \ldots, u_k$ permuted represents the same thing times ± 1, depending on the sign of the permutation; i.e., for any permutation σ of $\{0, 1, \ldots, k\}$,

$$[u_{\sigma(0)}, \ldots, u_{\sigma(k)}] = \mathrm{sign}(\sigma)[u_0, \ldots, u_k].$$

We use $\mathcal{C}_k = \mathcal{C}_k(\mathsf{K}_{\mathsf{abs}})$ to denote the set of *k-form of $\mathsf{K}_{\mathsf{abs}}$*, which are any formal linear combination of above symbols. We define $\partial_k \colon \mathcal{C}_k \to \mathcal{C}_{k-1}$ as the unique linear transformation with

$$\partial_k[u_0, \ldots, u_k] = \sum_{i=0}^{k} [u_0, \ldots, \widehat{u_i}, \ldots, u_k](-1)^i.$$

We easily verify that $\partial_{k-1}\partial_k = 0$ for all k, and we define the k-th homology group of $\mathsf{K}_{\mathsf{abs}}$ to be

$$H_k^{\mathrm{simp}}(\mathsf{K}_{\mathsf{abs}}) = \ker(\partial_k)/\mathrm{Image}(\partial_{k+1}).$$

For any k we define the *k-th Laplacian of $\mathsf{K}_{\mathsf{abs}}$* to be the operator:

$$\Delta_{\mathsf{K}_{\mathsf{abs}}}^k = \partial_{k-1}^{\mathrm{T}}\partial_k + \partial_k\partial_{k+1}^{\mathrm{T}}$$

where we take a basis for $\mathcal{C}_i(\mathsf{K}_{\mathsf{abs}})$ by choosing one orientation for each i-dimensional set $\{v_0, \ldots, v_i\} \in \mathsf{K}_{\mathsf{abs}}$. We define the space of *harmonic k-forms*, denoted $\mathcal{H}_k(\mathsf{K}_{\mathsf{abs}})$, to be the kernel of $\Delta_{\mathsf{K}_{\mathsf{abs}}}^k$.

The following fact is easy to prove.

Proposition 4.29. *With notation as above, any element $\mathcal{H}_k(\mathsf{K}_{\mathsf{abs}})$ lies in the kernel of ∂_k. The map $\mathcal{H}_k(\mathsf{K}_{\mathsf{abs}}) \to H_k^{\mathrm{simp}}(\mathsf{K}_{\mathsf{abs}})$ taking $\tau \in \mathcal{H}_k(\mathsf{K}_{\mathsf{abs}})$ to its class in $H_k^{\mathrm{simp}}(\mathsf{K}_{\mathsf{abs}}) = \ker(\partial_k)/\mathrm{Image}(\partial_{k+1})$ is an isomorphism.*

5. The Mayer-Vietoris Sequence for Graphs and Simplicial Complexes

There are many tools used to compute homology groups (or compare homology groups, etc.). The most basic one is a sort of "divide-and-conquer" tool, that lets you compute the homology group of an abstract simplicial complex by subdividing it into two parts. The case of graphs illustrates the basic principle.

Theorem 5.1 (Mayer-Vietoris)**.** *Let $G = (V, E)$ be a graph, and let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be subgraphs of G such that $G = G_1 \cup G_2$ (i.e., $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$). Then there is a chain complex*

$$\begin{tikzcd} 0 \arrow[r] & H_1(G_1 \cap G_2) \arrow[r, "``\mu_1\text{''}"] & H_1(G_1) \oplus H_1(G_2) \arrow[r, "``\nu_1\text{''}"] & H_1(G) \arrow[dll, "\delta"] \\ & H_0(G_1 \cap G_2) \arrow[r, "``\mu_0\text{''}"] & H_0(G_1) \oplus H_0(G_2) \arrow[r, "``\nu_0\text{''}"] & H_0(G) \arrow[r] & 0 \end{tikzcd} \tag{17}$$

that is exact*, meaning that the image of any arrow equals the kernel of the successive one.*

The labels above the arrows in (36) (e.g., "μ_1", δ, etc.) will be useful in our proof. This type of theorem is true in a number of different contexts, and is usually called the Mayer-Vietoris theorem.

Example 5.2. Let G_1 be the *path (graph)*, whose vertices in order are

$$A, A', B, B', C, C', D, D', E,$$

and let G_2 be the same on $A, A'', B, B'', C, C'', D, D'', E$. Any path graph has $\beta_0 = 1$ and $\beta_1 = 0$. [We will prove a picture in class.] However $G_1 \cap G_2$ is the graph of isolated vertices A, B, C, D, E, and hence has $\beta_0 = 5$ and $\beta_1 = 0$. Similarly $G_1 \cup G_2$ is connected has 4 independent cycles. In this case the exact sequence (36) becomes:

$$\begin{tikzcd} 0 \arrow[r] & H_1(G_1 \cap G_2) = 0 \arrow[r, "``\mu_1\text{''}"] & H_1(G_1) \oplus H_1(G_2) = 0 \arrow[r, "``\nu_1\text{''}"] & H_1(G) \simeq \mathbb{R}^4 \arrow[dll, "\delta"] \\ & H_0(G_1 \cap G_2) \simeq \mathbb{R}^5 \arrow[r, "``\mu_0\text{''}"] & H_0(G_1) \oplus H_0(G_2) \simeq \mathbb{R} \oplus \mathbb{R} \arrow[r, "``\nu_0\text{''}"] & H_0(G) \simeq \mathbb{R} \arrow[r] & 0 \end{tikzcd} \tag{18}$$

and hence, as vector spaces:

$$0 \to 0 \to 0 \to \mathbb{R}^4 \xrightarrow{\delta} \mathbb{R}^5 \to \mathbb{R}^2 \to \mathbb{R} \to 0.$$

Example 5.3. Of course, one could create a similar example where we go all the way to Z, where $\beta_0(G_1 \cap G_2) = 26$ and $\beta_1(G_1 \cup G_2) = 25$. The fact that $\beta_0(G_1 \cap G_2)$ and $\beta_1(G_1 \cup G_2)$ are very large and nearly equal is reflected in the exact sequence in Theorem 5.1

$$0 \to 0 \to 0 \to \mathbb{R}^{25} \xrightarrow{\delta} \mathbb{R}^{26} \to \mathbb{R}^2 \to \mathbb{R} \to 0.$$

Remark 5.4. It will take a bit of time to get used to the "exact sequences" that the Mayer-Vietoris theorem provides. One thing that is easy to see is that if

$$0 \xrightarrow{d_{m+1}} V_d \xrightarrow{d_m} \cdots \xrightarrow{d_1} V_0 \xrightarrow{d_0} 0$$

is an exact sequence (of finite dimensional vector spaces), then

$$\dim(V_0) - \dim(V_1) + \cdots + (-1)^d \dim(V_d) = 0. \tag{19}$$

Hence, for example, in the above two examples, we know that G_1, G_2 are paths, and hence $\beta_1(G_i) = 0$ and $\beta_0(G_i) = 1$ for $i = 1, 2$. Since $G = G_1 \cup G_2$ is connected, we have that $\beta_0(G) = 1$. If G_1, G_2 are any graphs satisfying the above, then the Mayer-Vietoris sequence gives a long exact sequence

$$0 \to 0 \to 0 \to H_1(G) \xrightarrow{\delta} H_0(G_1 \cap G_2) \to \mathbb{R}^2 \to \mathbb{R} \to 0,$$

and therefore (19) implies that

$$\dim\bigl(H_0(G_1 \cap G_2)\bigr) = \dim\bigl(H_1(G)\bigr),$$

i.e.,

$$\beta_0(G_1 \cap G_2) = \beta_1(G) + 1.$$

When we prove Theorem 5.1 we will see that all arrows in the above diagram as easy to construct, except δ; δ has various names, such as the "connecting map;" and the "snake lemma" is a way of conceptualizing the construction. However, once you see it here, you will know how to do this procedure (the "snake lemma") in general.

5.1. **Review of "Abstract" Linear Algebra.**

The more we work with homology groups and exact sequences, the better it will be to think in terms of "(abstract) vector spaces." At UBC you would have seen this in Math 223 (Honours Linear Algebra), and may have used textbooks like [J9̈4, Axl15]. Let us briefly explain this.

At UBC, the basic, one-term, linear algebra course, Math 221, focuses on systems of linear equations; one quickly sees the usefulness of working with

$$\mathbb{R}^n = \{(u_1, \ldots, u_n) \mid u_i \in \mathbb{R}\}$$

and its "addition law" and "scalar multiplication law"

$$(u_1, \ldots, u_n) + (w_1, \ldots, w_n) \stackrel{\text{def}}{=} (u_1 + w_1, \ldots, u_n + w_n), \quad \alpha(u_1, \ldots, u_n) \stackrel{\text{def}}{=} (\alpha u_1, \ldots, \alpha u_n).$$

$\mathbb{R}^n$ is often (sometimes?) called a "concrete vector space." A *linear transformation* $\mathcal{L}\colon \mathbb{R}^n \to \mathbb{R}^m$ is simply a map $\mathbf{u} \mapsto L\mathbf{u}$, where $\mathbf{u} \in \mathbb{R}^n$ is viewed as a column vector, and L is an $m \times n$ matrix with real entries (and $L\mathbf{u}$ is the usual matrix multiplication, interpreting a $1 \times n$ matrix as a "column vector" lying in $\mathbb{R}^n$) The *kernel* of *nullspace* of $\mathcal{L}$, i.e., $\ker(\mathcal{L}) \stackrel{\text{def}}{=} \{\mathbf{u} \mid \mathcal{L}\mathbf{u} = \mathbf{0}\}$ is also an important concept, and this is a subspace of $\mathbb{R}^n$ and also (often) called a "concrete vector space." The image of L, $\mathrm{Image}(\mathcal{L}) \stackrel{\text{def}}{=} \{\mathcal{L}\mathbf{u}\}$ is subspace of $\mathbb{R}^m$, again (often) called a "concrete vector space."

At this point we know that the $\mathrm{coker}(\mathcal{L})$ is the *quotient space* $\mathbb{R}^m/\mathrm{Image}(\mathcal{L})$ (see Subsection 4.7.3, which we needed to define $H_0(G) \stackrel{\text{def}}{=} \mathrm{coker}(\partial_1)$). Since $\mathrm{coker}(\mathcal{L})$

isn't canonically (or naturally[6]) a subspace of $\mathbb{R}^m$[7] (or of $\mathbb{R}^{m'}$ for any m'), most authors wouldn't call such a quotient space in $\mathbb{R}^n$ a "concrete vector space." Instead, we define a "vector space" (or sometimes "abstract vector space") any set U with an "addition law" and "scalar multiplication law" that satisfies a set of axioms that hold for $\mathbb{R}^m$.

The basic set of tools we need to work with "abstract" vector spaces — such as to define linear independence, bases, etc. — is really no different than what is needed for $\mathbb{R}^n$; it just requires a bit of "getting used to."[8]

Until now, many vector spaces we have seen are "concrete" vector spaces: for example, to define the homology groups of a graph $G = (V, E)$, for $i = 0, 1$, we defined the i-forms (or i-chains) on G to be a space $\mathcal{C}_i(G)$; however, we can identify $\mathcal{C}_0(G)$ with $\mathbb{R}^{|V|}$ (after ordering V) and identify $\mathcal{C}_1(G)$ with $\mathbb{R}^{|E|}$ (after orienting each edge and ordering the oriented edges). Notice that these identifications involve some ad hoc choices, and are not canonical. We also defined

$$\partial_1 \colon \mathcal{C}_1(G) \to \mathcal{C}_0(G),$$

but for calculations we often wrote ∂_1 as a matrix. Then $H_1(G) \stackrel{\text{def}}{=} \ker(\partial_1)$ can be viewed a subspace of $\mathbb{R}^{|E|}$, and $H_0(G) \stackrel{\text{def}}{=} \operatorname{coker}(\partial_1)$ is a quotient (and not really a "concrete" vector space).

Most CPSC 531F have seen linear algebra (or "abstract linear algebra") before, but may not have seen any applications of it.[9]. In Appendix D we will gather a summary of what we need in these notes; we also refer to the textbooks [J̈94, Axl15], or Chapter 0 of *Matrix Analysis* by Horn and Johnson ([HJ13], or earlier editions, [HJ85, HJ90]).

5.2. **Chains and Exact Sequences.** More generally, a *chain of vector spaces* is any sequence $(\mathbf{V}, \mathbf{d})$ of vector spaces and maps

$$\cdots \xrightarrow{d_2} V_1 \xrightarrow{d_1} V_0 \xrightarrow{d_0} V_{-1} \xrightarrow{d_{-1}} \cdots \tag{20}$$

such that $d_i d_{i+1} = 0$ for all i; we say that (20) is *exact in position i* if $\ker(d_i) = \operatorname{Image}(d_{i+1})$, and we say that (20) is *exact* if it is exact in every position.

More generally, since $d_i d_{i+1} = 0$ for all i, we have

$$\operatorname{Image}(d_{i+1}) \subset \ker(d_i),$$

[6]The term "canonical" and "canonically" are extremely important ideas, but are not mathematically precise; in the context of vector spaces, a construction is "canonical" if it can be defined without making some ad hoc choices of bases for the vector spaces involved; the general meaning of "canonical" is a construction "without ad hoc choices," especially when such choices change the actual structure we construct. By contrast, the term "natural" is a completely precise (and important) term, and is an adjective for a *functor* constructed in the context of category theory. In the context of vector spaces, a "canonical" construction is almost ways "natural." It is possible that some authors differ from us on the meaning of "canonical" and "natural"...

[7]In Subsection 4.7.3 we mentioned that for $\mathbb{R}^n$ one can identify $\mathbb{R}^n/U$ with $U^\perp$, but this doesn't work if $\mathbb{R}$ is replaced with an arbitrary field, and it is a conceptual mistake to view $\mathbb{R}^n/U$ and $U^\perp$ as the same thing.

[8]In class I may mention the famous quote of John von Neumann responding to a physicist who complained of not understanding the method of characteristics.

[9]Indeed, in a lot of computer science, you don't really need to know about "abstract" vector spaces.

and we define the *i-th homology group of* $(\mathbf{V},\mathbf{d})$ to be

$$H_i(\mathbf{V},\mathbf{d}) \stackrel{\mathrm{def}}{=} \ker(d_i)/\mathrm{Image}(d_{i+1}), \tag{21}$$

and the *i-th Betti number of* $(\mathbf{V},\mathbf{d})$ to be

$$\beta_i(\mathbf{V},\mathbf{d}) = \dim\bigl(H_i(\mathbf{V},\mathbf{d})\bigr). \tag{22}$$

To say that $(\mathbf{V},\mathbf{d})$ is *exact in position* i is just to say that $H_i(\mathbf{V},\mathbf{d})=0$, and to say that it is *exact* means $H_i(\mathbf{V},\mathbf{d})=0$ for all i).

We will often work with finite chains, i.e., chains (20) where $V_i=0$ both for i sufficiently small and i sufficiently large. In this case we write the chain $(\mathbf{V},\mathbf{d})$ as:

$$0 \xrightarrow{d_{m+1}} V_d \xrightarrow{d_m} \cdots \xrightarrow{d_1} V_0 \xrightarrow{d_0} 0,$$

implying that $V_i=0$ for $i\ge m+1$ and $i\le -1$. If each $\beta_i=\dim(H_i(\mathbf{V},\mathbf{d}))$ is finite, then we define the *Euler characteristic of* $(\mathbf{V},\mathbf{d})$ to be

$$\chi(\mathbf{V},\mathbf{d}) = \beta_0-\beta_1+\cdots+(-1)^d\beta_d. \tag{23}$$

In Exercise A.17 we will show that if each V_i is finite dimensional, then

$$\chi(\mathbf{V},\mathbf{d}) = \dim(V_0)-\dim(V_1)+\cdots+(-1)^d\dim(V_d).$$

5.3. **Exact Sequences: Some Simple Examples.** It takes a little while to get used to exact sequences, but they are a sort of "counting" for vector spaces. For example, if S_1,S_2 are finite sets, then inclusion-exclusion implies

$$|S_1|+|S_2| = |S_1\cup S_2|+|S_1\cap S_2|.$$

This equality can be expressed in terms an exact sequence as follows.

First, for a set, S, we let $\mathbb{R}[S]$ be (as usual) the set of $\mathbb{R}$-linear formal combinations of elements of S (Definition 4.1. For example, if $S=\{A,B,C\}$, then $\mathbb{R}[S]$ includes:

$$0,\ 3A,\ (-3)A+\sqrt{2}C,\ B-(12)C, 3A+4B+5C.$$

By convention, a formal sum can omit elements of S. In this way, if $S\subset T$, we can view $\mathbb{R}[S]\subset\mathbb{R}[T]$ (or, at least, there is a natural inclusion of $\mathbb{R}[S]$ into $\mathbb{R}[T]$[10]).

Next, let $S_1,S_2\subset S$ be sets. We will build an exact sequence:

$$0\to\mathbb{R}[S_1\cap S_2] \xrightarrow{\ \mu\ } \mathbb{R}[S_1]\oplus\mathbb{R}[S_2] \xrightarrow{\ \nu\ } \mathbb{R}[S_1\cup S_2]\to 0. \tag{24}$$

There is some flexibility in building μ and ν, but let us choose a simple way. Since $S_1\cap S_2$ is a subset of S_1 and of S_2, we have

$$\mathbb{R}[S_1\cap S_2]\subset\mathbb{R}[S_1],\quad \mathbb{R}[S_1\cap S_2]\subset\mathbb{R}[S_2],$$

and we will let μ be the map given by $\mu(\tau)=(\tau,\tau)$. Similarly, both $\mathbb{R}[S_1],\mathbb{R}[S_2]$ lie in (or are naturally viewed as a subspace of) $\mathbb{R}[S_1\cup S_2]$. So set $\nu(\sigma_1,\sigma_2)=\sigma_1-\sigma_2$.

Example 5.5. Let $S_1=\{A,B,C,D\}$ and $S_2=\{D,E\}$. Then $\tau=4D\in\mathbb{R}[S_1\cap S_2]$, and $\mu(\tau)=(4D,4D)\in\mathbb{R}[S_1]\oplus\mathbb{R}[S_2]$. As examples:

$$\nu(3A+2C,5D-E)=3A+2C-(5D-E)\in\mathbb{R}[S_1\cup S_2],$$

and

$$\nu(\mu(4D))=\nu(4D,4D)=4D-4D=0\in\mathbb{R}[S_1\cup S_2].$$

[10]Technically if $t\in T\setminus S$, then $0=4t-4t$, but $4t-4t$ doesn't lie in $\mathbb{R}[S]$; so $0\in\mathbb{R}[T]$ is technically a larger equivalence class of expressions than is $0\in\mathbb{R}[S]$.

Continuing on with the general situation for (24): for any $\tau \in \mathbb{R}[S_1 \cap S_2]$,

$$\nu\mu(\tau) = \nu(\tau, \tau) = \tau - \tau = 0.$$

EXERCISE: Check that (24) is exact at each position. Show that exactness in position $\mathbb{R}[S_1 \cap S_2]$ holds iff μ is injective; similarly, exactness in position $\mathbb{R}[S_1 \cup S_2]$ holds iff ν is surjective.

Definition 5.6. By a *short exact sequence* we mean a chain of vector spaces:

$$0 \to V_2 \xrightarrow{d_2} V_1 \xrightarrow{d_1} V_0 \to 0. \tag{25}$$

EXERCISE: Show that (25) is exact in position 2 (i.e., at V_2) iff d_2 is injective; similarly, exactness in position 0 (i.e., at V_0) holds iff d_1 is surjective.

EXERCISE: Show that if $U_1, U_2 \subset U$ are subspaces of an $\mathbb{R}$-vector space U, and $U_1 + U_2$ is their span, then give a short exact sequence

$$0 \to U_1 \cap U_2 \xrightarrow{\mu} U_1 \oplus U_2 \xrightarrow{\nu} U_1 + U_2 \to 0,$$

constructed similarly to (24).

Remark 5.7. If $G_1 \amalg G_2$ denotes the disjoint union[11] of G_1 and G_2, then

$$H_i(G_1 \amalg G_2) \simeq H_i(G_1) \oplus H_i(G_2).$$

Hence you could also write the long exact sequence in Theorem 5.1 using $H_i(G_1 \amalg G_2)$ instead of $H_i(G_1) \oplus H_i(G_2)$ everywhere.

5.4. **Step One of the Proof of Theorem 5.1: The Commutative Diagram.** The vertex set of $G_1 \cap G_2$ is, by definition $V_1 \cap V_2$, and $V = V_1 \cup V_2$. Hence we have a short exact sequence:

$$0 \to \mathbb{R}[V_1 \cap V_2] \xrightarrow{\mu_0} \mathbb{R}[V_1] \oplus \mathbb{R}[V_2] \xrightarrow{\nu_0} \mathbb{R}[V] \to 0.$$

where

$$\mu_0(\tau) = (\tau, \tau), \quad \nu_0(\sigma_1, \sigma_2) = \sigma_1.$$

This is therefore a short exact sequence on 0-forms:

$$0 \to \mathcal{C}_0(G_1 \cap G_2) \xrightarrow{\mu_0} \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2) \xrightarrow{\nu_0} \mathcal{C}_0(G) \to 0.$$

We similarly get an exact sequence on 1-forms. Combining these maps and the ∂_1 maps, we get the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}_1(G_1 \cap G_2) \arrow[r, "\mu_1"] \arrow[d, "\partial_1"'] & \mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2) \arrow[r, "\nu_1"] \arrow[d, "\partial_1 \oplus \partial_1"'] & \mathcal{C}_1(G) \arrow[r] \arrow[d, "\partial_1"'] & 0 \\
0 \arrow[r] & \mathcal{C}_0(G_1 \cap G_2) \arrow[r, "\mu_0"] & \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2) \arrow[r, "\nu_0"] & \mathcal{C}_0(G) \arrow[r] & 0
\end{tikzcd}, \tag{26}$$

where the ∂_1 are all taken in their respective graphs (but all of them are the maps given by $\partial_1([v, v']) = [v'] - [v]$, so, in a sense, they are all the same map). We note that by our choices of $\mu_0, \mu_1, \nu_0, \nu_1$, this diagram "commutes" in the sense that the composition of any horizontal arrow followed by a vertical arrow (starting in

[11]The *disjoint union* is a limit and not uniquely defined. You build this from the disjoint union of sets, S, T, which can be taken to be $S \times \{1\} \cup T \times \{2\}$ (hence this gives the union of two disjoint sets, one isomorphic to S and the other to T, even if S, T have a non-zero intersection as sets). Then one defines $G_1 \amalg G_2$ as the graph with vertex set $V_1 \amalg V_2$ and edge set $E_1 \amalg E_2$.

a nonzero vector space on the top row) equals the same where we first take the vertical arrow followed by the horizontal arrow; for example

$$\bigl((\partial_1 \oplus \partial_1)\mu_1\bigr)(\tau) = (\partial_1\tau, \partial_1\tau) = (\mu_0\partial_1)(\tau).$$

5.5. **Step Two of the Proof of Theorem 5.1: The First Two Maps.** (given in class) to prove Theorem 5.1.

For example, to construct the map

$$H_1(G_1 \cap G_2) \to H_1(G_1) \oplus H_1(G_2),$$

we take any element $\tau \in H_1(G_1 \cap G_2)$, i.e., $\tau \in \mathcal{C}_1(G_1 \cap G_2)$ with $\partial_1\tau = 0$, and easily see that

$$(\partial_1 \oplus \partial_1)\mu_1\tau = (\partial_1\tau, \partial_1\tau) = (0, 0),$$

and hence μ_1 takes $H_1(G_1 \cap G_2)$ to $H_1(G_1) \oplus H_1(G_2)$.

We similarly construct the maps:

$$H_1(G_1) \oplus H_1(G_2) \to H_1(G_1 \cap G_2).$$

5.6. **Step Three of the Proof of Theorem 5.1: The Last Two Maps.** Next we describe how the map

$$\mathcal{C}_0(G_1 \cap G_2) \xrightarrow{\mu_0} \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2)$$

induces a map

$$H_0(G_1 \cap G_2) \xrightarrow{\text{“}\mu_0\text{”}} H_0(G_1) \oplus H_0(G_2).$$

The map “μ_0” should come from the segment of (26):

$$\begin{tikzcd} \mathcal{C}_1(G_1 \cap G_2) \arrow[r, "\mu_1"] \arrow[d, "\partial_1"'] & \mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2) \arrow[d, "\partial_1 \oplus \partial_1"'] \\ \mathcal{C}_0(G_1 \cap G_2) \arrow[r, "\mu_0"] & \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2) \end{tikzcd} , \tag{27}$$

and we need to remember that

$$H_0(G_1 \cap G_2) = \mathcal{C}_0(G_1 \cap G_2)/I_{12}, \text{ where } I_{12} = \mathrm{Image}\bigl(\partial_1(G_1 \cap G_2)\bigr) = \partial_1\bigl(\mathcal{C}_1(G_1 \cap G_2)\bigr),$$

and similarly, for $j = 1, 2$,

$$H_0(G_j) = \mathcal{C}_0(G_i)/I_j, \text{ where } I_j = \mathrm{Image}\bigl(\partial_1(G_j)\bigr) = \partial_1\bigl(\mathcal{C}_1(G_j)\bigr).$$

Since $\mu_0(\tau) = (\tau, \tau)$ in (27), the map “μ_0” should map $\tau \in H_0(G_1 \cap G_2)$ to “(τ, τ),” provided that this makes sense (i.e., is well defined).

Recall that literally, $\mathcal{C}_0(G_1 \cap G_2) = \mathbb{R}[V_1 \cap V_2]$, where V_1, V_2 are the vertex sets of G_1, G_2, and for $j = 1, 2$, $\mathcal{C}_0(G_j) = \mathbb{R}[V_j]$, which for $j = 1, 2$ gives an “inclusion map”

$$\iota_j \colon \mathcal{C}_0(G_1 \cap G_2) \to \mathcal{C}_0(G_j).$$

Hence we need to make sure that it extends to a linear map

$$\text{“}\iota_j\text{”} \colon \mathcal{C}_0(G_1 \cap G_2)/I_{12} \to \mathcal{C}_0(G_j)/I_j. \tag{28}$$

This will only work if $\iota_j(I_{12}) \subset I_j$ — the more general principle is this: if $U_1 \subset U$ and $W_1 \subset W$ are subspaces of vector spaces, then a linear map $\mathcal{L} \colon U \to W$ extends to a map $\mathcal{L} \colon U/U_1 \to W/W_1$ iff $\mathcal{L}(U_1) \subset W_1$. Let us give some examples.

5.7. **Aside on Maps of Quotient Spaces, and their Approach via "Diagram Chasing".** Recall the definition of quotient spaces of vector spaces Subsection 4.7.3. In this Subsection we explain that if $U_1 \subset U$ and $W_1 \subset W$ are subspaces of vector spaces, then a linear map $\mathcal{L} \colon U \to W$ extends to a map $\mathcal{L} \colon U/U_1 \to W/W_1$ iff $\mathcal{L}(U_1) \subset W_1$.

At this point in CPSC 531F, we gave reviewed the definition of quotient spaces, maps between quotient spaces, and gave a number of examples; see Section D.8 for these examples in detail. Then we gave a "diagram chasing" view of this.

For examples of maps between quotient spaces, it is helpful to keep the following examples in mind: first, in modular arithmetic:

(1) the identity map $\mathcal{L} \colon \mathbb{Z} \to \mathbb{Z}$ extends to a map $\mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$;
(2) the identity map $\mathcal{L} \colon \mathbb{Z} \to \mathbb{Z}$ does not extend to a map $\mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$; but
(3) the map $\mathcal{L} \colon \mathbb{Z} \to \mathbb{Z}$ given by $\mathcal{L}(u) = 2u$ does extend to a map $\mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$, since $\mathcal{L}(3\mathbb{Z}) \subset 6\mathbb{Z}$, or, by example
$$\mathcal{L}(1 + 3\mathbb{Z}) = 2 + 6\mathbb{Z} \subset \mathbb{Z}/6\mathbb{Z}.$$

Second, for vector spaces:

(1) Let $U = W = \mathbb{F}^2$, $U_1 = \{(x,x) | x \in \mathbb{R}\}$, and $W_1 = \{(x,-2x) | x \in \mathbb{R}\}$. Then the identity map $\mathcal{L} \colon \mathbb{F}^2 \to \mathbb{F}^2$ does not extend to a map $\mathbb{F}^2/U_1 \to \mathbb{F}^2/W_1$, since $\mathcal{L}(U_1)$ is not a subset of W_1.
(2) However, the map $\mathcal{L} \colon \mathbb{F}^2 \to \mathbb{F}^2$ given by $\mathcal{L}(x,y) = (x+y, -5x+y)$ does extend to a map $\mathbb{F}^2/U_1 \to \mathbb{F}^2/W_1$, since $\mathcal{L}(U_1) \subset \mathcal{L}(W_1)$, since $\mathcal{L}(x,x) = (2x, -4x) \in W_1$. For example, the three "coset representatives" of the coset $(0,3) + U_1$,
$$(0,3), (1,4), (-1,2) \in (0,3) + U_1 = \{(x, 3+x) | x \in \mathbb{R}\},$$
are respectively mapped to
$$(3,3), (5,-1), (1,7) \in (3,3) + U_2 = \{(3+x, 3-2x)\ x \in \mathbb{R}\},$$
hence each is a "coset representative" of the coset $(3,3) + U_1$.

To express the last example "in the spirit of diagram chasing," we write the diagram:

$$\begin{tikzcd} U_1 \arrow[d] & W_1 \arrow[d] \\ U \arrow[r, "\mathcal{L}"] & W \end{tikzcd} \tag{29}$$

and we notice that if $\mathcal{L}(U_1) \subset W_1$, then we get a (unique) map $\mathcal{L}' \colon U_1 \to W_1$, namely the restriction of $\mathcal{L}$ to U_1, that gives us a commutative diagram:

$$\begin{tikzcd} U_1 \arrow[r, dashed, "\mathcal{L}'"] \arrow[d] & W_1 \arrow[d] \\ U \arrow[r, "\mathcal{L}"] & W \end{tikzcd}$$

(where the newly constructed map $\mathcal{L}' \colon U_1 \to W_1$ is indicated in a dashed line). Now we claim that this gives us a map $\mathcal{L}'' \colon U/U_1 \to W/W_1$:

$$\begin{tikzcd}
U_1 \arrow[r, "\mathcal{L}'"] \arrow[d] & W_1 \arrow[d] \\
U \arrow[r, "\mathcal{L}"] \arrow[d] & W \arrow[d] \\
U/U_1 \arrow[r, dashed, "\mathcal{L}''"] & W/W_1
\end{tikzcd}$$

where, again, the dashed arrow is the map $\mathcal{L}'' \colon U/U_1 \to W/W_1$ that we need to build. The details of building $\mathcal{L}''$ is as follows: pick a $\beta \in U/U_1$, and let us detemine $\mathcal{L}''(\beta)$: to do so, we pick a $\beta_1 \in U$ mapping to β; then $\alpha_1 \in \mathcal{L}(\beta_1) \in W$ and this maps to an element $\alpha_1' \in W/W_1$:

$$\begin{tikzcd}
 & U_1 \arrow[r, "\mathcal{L}'"] \arrow[d] & W_1 \arrow[d] & \\
\beta_1 \in \arrow[d, mapsto] \arrow[rrr, mapsto, bend left] & U \arrow[r, "\mathcal{L}"] \arrow[d] & W \arrow[d] & \ni \alpha_1 \arrow[d, mapsto] \\
\beta \in & U/U_1 \arrow[r, dashed, "\mathcal{L}''"] & W/W_1 & \ni \alpha_1'
\end{tikzcd}$$

Now we want to make sure that the $\alpha_1' \in W/W_1$ we get doesn't depend on the choice of β_1: so say that we choose another $\beta_2 \in U$ mapping to $\beta \in U/U_1$: hence $\beta_1 - \beta_2$ maps to 0 in U/U_1; by exactness we have $\sigma \in U_1$ maps to $\beta_1 - \beta_2$ (indeed, by definition of U/U_1, $\beta_1 - \beta_2 \in U_1$); let $\tau = \mathcal{L}'(\sigma)$; by commutativity τ is taken to $\tau_1 \in W$ that must equal $\alpha_1 - \alpha_2$; hence, if $\alpha_1' \in W/W_1$ is as before, and $\alpha_2' \in W/W_1$ is the analog for β_2, $\alpha_1' - \alpha_2'$, is the image of τ under two vertical arrows. Now notice that applying two vertical arrows takes W_1 to 0 in W/W_1; hence $\alpha_1' - \alpha_2' = 0$ in W/W_1, and hence $\alpha_1' = \alpha_2$ in W/W_1. Hence the choice of β_1 for β, used to define $\alpha_1' = \mathcal{L}(\beta_1)$ in W/W_1, does not depend on the choice of β_1.

$$\begin{tikzcd}
\sigma \in \arrow[d, mapsto] \arrow[rrr, mapsto, bend left] & U_1 \arrow[r, "\mathcal{L}'"] \arrow[d] & W_1 \arrow[d] & \ni \tau = \mathcal{L}'(\sigma) \arrow[d, mapsto] \\
\beta_1 - \beta_2 \in \arrow[d, mapsto] \arrow[rrr, mapsto, bend left] & U \arrow[r, "\mathcal{L}"] \arrow[d] & W \arrow[d] & \ni \tau_1 = \alpha_1 - \alpha_2 \arrow[d, mapsto] \\
0 \in & U/U_1 \arrow[r, dashed, "\mathcal{L}''"] & W/W_1 & \ni 0 = \alpha_1' - \alpha_2'
\end{tikzcd}$$

The exact same "diagram chase" proves the more general result:

Proposition 5.8. *Consider any commutative diagram:*

$$\begin{tikzcd} U_1 \arrow[r, "\mathcal{L}_1"] \arrow[d, "d_U"'] & W_1 \arrow[d, "d_W"] \\ U_0 \arrow[r, "\mathcal{L}_0"] & W_0 \end{tikzcd}$$

Let, as usual, $\mathrm{coker}(d_U) = U_0/\mathrm{Image}(d_U)$*, and let* $U_0 \to \mathrm{coker}(d_U)$ *be the canonical map taking* $u \in U_0$ *to its image in the quotient space (i.e., to* $u + \mathrm{Image}(d_U)$*). Doing similarly for* d_W*, we get a diagram:*

$$\begin{tikzcd} U_1 \arrow[r, "\mathcal{L}_1"] \arrow[d, "d_U"'] & W_1 \arrow[d, "d_W"] \\ U_0 \arrow[r, "\mathcal{L}_0"] \arrow[d] & W_0 \arrow[d] \\ \mathrm{coker}(d_U) & \mathrm{coker}(d_W) \end{tikzcd}$$

Then there is a unique map $\mathcal{L}''\colon \mathrm{coker}(d_U) \to \mathrm{coker}(d_W)$ *giving a commutative diagram:*

$$\begin{tikzcd} U_1 \arrow[r, "\mathcal{L}_1"] \arrow[d, "d_U"'] & W_1 \arrow[d, "d_W"] \\ U_0 \arrow[r, "\mathcal{L}_0"] \arrow[d] & W_0 \arrow[d] \\ \mathrm{coker}(d_U) \arrow[r, dashed, "\mathcal{L}''"] & \mathrm{coker}(d_W) \end{tikzcd}$$

5.8. **The End of Step Three of the Proof of Theorem 5.1.** So to extend the map

$$\mu_0\colon \mathcal{C}_0(G_1 \cap G_2) \to \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2)$$

to a map

$$H_0(G_1 \cap G_2) \to H_0(G_1) \oplus H_0(G_2),$$

we need to verify that if $\tau, \tau' \in \mathcal{C}_0(G_1 \cap G_2)$ satisfy $\tau - \tau' \in \mathrm{Image}(\partial_1)$, then $\mu_0(\tau)$ and $\mu_0(\tau')$ are in the same equivalence class (or coset) in $H_0(G_1) \oplus H_0(G_2)$. However, since $\tau - \tau' \in \mathrm{Image}(\partial_1)$, we have $\partial_1\sigma = \tau - \tau'$ for some $\sigma \in \mathcal{C}_1(G_1 \cap G_2)$, and hence $\mu_0(\tau - \tau') = \partial_1(\mu_1(\sigma))$, where this ∂_1 refers to the middle verticle arrow in (26). In class we show that you are really using the commutativity of the diagram (26).

We similarly construct the last map in (26),

$$H_0(G_1) \oplus H_0(G_2) \to H_0(G_1 \cap G_2),$$

because the second to last map, built just above, was based on the commutativity of the horizontal and vertical arrows and nothing particular about these arrows.

5.9. **Step Four of the Proof of Theorem 5.1.** The real heart of the argument is to construct the map

$$H_1(G_1 \cup G_2) \xrightarrow{\delta} H_0(G_1 \cap G_2),$$

which is the more remarkable "diagram chasing" argument: for $\beta \in H_1(G_1 \cup G_2)$ we determine $\delta(\beta)$ as follows:

(1) By definition, $\beta \in H_1(G)$, and hence $\beta \in \mathcal{C}_1(G)$ and satisfies $\partial_1 \beta = 0$:

$$\begin{tikzcd}
 & & & \beta \arrow[dd, mapsto, dashed, bend left] & \\
0 \arrow[r] & \mathcal{C}_1(G_1 \cap G_2) \arrow[r, "\mu_1"] \arrow[d, "\partial_1"'] & \mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2) \arrow[r, "\nu_1"] \arrow[d, "\partial_1 \oplus \partial_1"'] & \mathcal{C}_1(G) \arrow[r] \arrow[d, "\partial_1"'] & 0 \\
0 \arrow[r] & \mathcal{C}_0(G_1 \cap G_2) \arrow[r, "\mu_0"] & \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2) \arrow[r, "\nu_0"] & \mathcal{C}_0(G) \arrow[r] & 0 \\
 & & & \partial_1 \beta = 0 &
\end{tikzcd}$$

(2) Since ν_1 is surjective, we have $\nu_1(\sigma_1, \sigma_2) = \beta$ for some $(\sigma_1, \sigma_2) \in \mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2)$:

$$\begin{tikzcd}
 & & (\sigma_1, \sigma_2) \arrow[r, mapsto, dashed] & \beta = \nu_1(\sigma_1, \sigma_2) \arrow[ddd, mapsto, bend left] & \\
0 \arrow[r] & \mathcal{C}_1(G_1 \cap G_2) \arrow[r, "\mu_1"] \arrow[d, "\partial_1"'] & \mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2) \arrow[r, "\nu_1"] \arrow[d, "\partial_1 \oplus \partial_1"'] & \mathcal{C}_1(G) \arrow[r] \arrow[d, "\partial_1"'] & 0 \\
0 \arrow[r] & \mathcal{C}_0(G_1 \cap G_2) \arrow[r, "\mu_0"] & \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2) \arrow[r, "\nu_0"] & \mathcal{C}_0(G) \arrow[r] & 0 \\
 & & & 0 &
\end{tikzcd}$$

(3) Since the diagram above is commutative, we have $(\partial_1 \sigma_1, \partial_1 \sigma_2)$ is mapped by ν_0 to 0.

$$\begin{tikzcd}
 & & (\sigma_1, \sigma_2) \arrow[r, mapsto] \arrow[ddd, mapsto, dashed, bend left] & \beta \arrow[ddd, mapsto, bend left] & \\
0 \arrow[r] & \mathcal{C}_1(G_1 \cap G_2) \arrow[r, "\mu_1"] \arrow[d, "\partial_1"'] & \mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2) \arrow[r, "\nu_1"] \arrow[d, "\partial_1 \oplus \partial_1"'] & \mathcal{C}_1(G) \arrow[r] \arrow[d, "\partial_1"'] & 0 \\
0 \arrow[r] & \mathcal{C}_0(G_1 \cap G_2) \arrow[r, "\mu_0"] & \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2) \arrow[r, "\nu_0"] & \mathcal{C}_0(G) \arrow[r] & 0 \\
 & & (\partial_1 \sigma_1, \partial_1 \sigma_2) \arrow[r, mapsto, dashed] & 0 &
\end{tikzcd}$$

(4) Hence, by short exactness (in the middle position) of the bottom row of the diagram above, since $(\partial_1 \sigma_1, \partial_1 \sigma_2)$ maps to 0 under ν_0, there is a $\tau \in \mathcal{C}_0(G_1 \cap G_2)$ with $\mu_0(\tau) = (\partial_1 \sigma_1, \partial_1 \sigma_2)$; by short exactness (in the $\mathcal{C}_0(G_1 \cap G_2)$ position) of the bottom row, the map μ_0 is injective, and hence this τ is

uniquely determined by $(\partial_1\sigma_1, \partial_1\sigma_2)$:

$$\begin{tikzcd}
 & & (\sigma_1,\sigma_2) \arrow[r, mapsto] \arrow[ddd, bend left, mapsto] & \beta \arrow[ddd, bend left, mapsto] & \\
0 \arrow[r] & \mathcal{C}_1(G_1\cap G_2) \arrow[r, "\mu_1"] \arrow[d, "\partial_1"'] & \mathcal{C}_1(G_1)\oplus\mathcal{C}_1(G_2) \arrow[r, "\nu_1"] \arrow[d, "\partial_1\oplus\partial_1"'] & \mathcal{C}_1(G) \arrow[r] \arrow[d, "\partial_1"'] & 0 \\
0 \arrow[r] & \mathcal{C}_0(G_1\cap G_2) \arrow[r, "\mu_0"] & \mathcal{C}_0(G_1)\oplus\mathcal{C}_0(G_2) \arrow[r, "\nu_0"] & \mathcal{C}_0(G) \arrow[r] & 0 \\
 & \tau \arrow[r, mapsto, dashed] & \mu_0\tau = (\partial_1\sigma_1,\partial_1\sigma_2) \arrow[r, mapsto] & 0 & \\
 & (\tau \text{ is unique}) & & &
\end{tikzcd}$$

(5) The way we found τ above depended on a choice of (σ_1, σ_2) such that $\nu_1(\sigma_1, \sigma_2) = \beta$. However, we claim that the equivalence class (or coset) of τ in $H_0(G_1 \cap G_2)$ does not depend on the choice of (σ_1, σ_2). To prove this, say that (σ_1', σ_2') satisfies $\nu_1(\sigma_1', \sigma_2') = \beta$, and say that τ' is the element of $\mathcal{C}_0(G_1 \cap G_2)$ that would result by the analogous process or "diagram chase":

$$\begin{tikzcd}
 & & (\sigma_1',\sigma_2') \arrow[r, mapsto] \arrow[ddd, bend left, mapsto] & \beta \arrow[ddd, bend left, mapsto] & \\
0 \arrow[r] & \mathcal{C}_1(G_1\cap G_2) \arrow[r, "\mu_1"] \arrow[d, "\partial_1"'] & \mathcal{C}_1(G_1)\oplus\mathcal{C}_1(G_2) \arrow[r, "\nu_1"] \arrow[d, "\partial_1\oplus\partial_1"'] & \mathcal{C}_1(G) \arrow[r] \arrow[d, "\partial_1"'] & 0 \\
0 \arrow[r] & \mathcal{C}_0(G_1\cap G_2) \arrow[r, "\mu_0"] & \mathcal{C}_0(G_1)\oplus\mathcal{C}_0(G_2) \arrow[r, "\nu_0"] & \mathcal{C}_0(G) \arrow[r] & 0 \\
 & \tau' \arrow[r, mapsto] & \mu_0\tau' = (\partial_1\sigma_1',\partial_1\sigma_2') \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

Then we now prove that τ' and τ are in the same class (or coset) in $H_0(G_1 \cap G_2)$: indeed, by linearity we can subtract the digram for τ' from that for τ:

$$\begin{tikzcd}
 & & (\sigma_1,\sigma_2)-(\sigma_1',\sigma_2') \arrow[r, mapsto] \arrow[ddd, bend left, mapsto] & \beta-\beta & \\
0 \arrow[r] & \mathcal{C}_1(G_1\cap G_2) \arrow[r, "\mu_1"] \arrow[d, "\partial_1"'] & \mathcal{C}_1(G_1)\oplus\mathcal{C}_1(G_2) \arrow[r, "\nu_1"] \arrow[d, "\partial_1\oplus\partial_1"'] & \mathcal{C}_1(G) \arrow[r] \arrow[d, "\partial_1"'] & 0 \\
0 \arrow[r] & \mathcal{C}_0(G_1\cap G_2) \arrow[r, "\mu_0"] & \mathcal{C}_0(G_1)\oplus\mathcal{C}_0(G_2) \arrow[r, "\nu_0"] & \mathcal{C}_0(G) \arrow[r] & 0 \\
 & \tau-\tau' \arrow[r, mapsto] & (\partial_1\sigma_1,\partial_1\sigma_2)-(\partial_1\sigma_1',\partial_1\sigma_2') & &
\end{tikzcd} \tag{30}$$

In particular

$$\mu_0(\tau - \tau') = (\partial_1\sigma_1, \partial_1\sigma_2) - (\partial_1\sigma_1', \partial_1\sigma_2'). \tag{31}$$

Since $\beta - \beta = 0$ (in the upper right or $\mathcal{C}_1(G)$ position), we have ν_1 takes $(\sigma_1, \sigma_2) - (\sigma_1', \sigma_2')$ to 0, and by exactness of the top row we have that

$$(\sigma_1, \sigma_2) - (\sigma_1', \sigma_2') = \mu_1(\alpha) \tag{32}$$

for some $\alpha \in \mathcal{C}_1(G_1 \cap G_2)$:

$$
\begin{tikzcd}
 & \alpha \arrow[r, mapsto, dashed] & (\sigma_1,\sigma_2)-(\sigma_1',\sigma_2') \arrow[r, mapsto] & \beta-\beta=0 & \\
0 \arrow[r] & \mathcal{C}_1(G_1\cap G_2) \arrow[r, "\mu_1"] \arrow[d, "\partial_1"'] & \mathcal{C}_1(G_1)\oplus\mathcal{C}_1(G_2) \arrow[r, "\nu_1"] \arrow[d, "\partial_1\oplus\partial_1"'] & \mathcal{C}_1(G) \arrow[r] \arrow[d, "\partial_1"'] & 0 \\
0 \arrow[r] & \mathcal{C}_0(G_1\cap G_2) \arrow[r, "\mu_0"] & \mathcal{C}_0(G_1)\oplus\mathcal{C}_0(G_2) \arrow[r, "\nu_0"] & \mathcal{C}_0(G) \arrow[r] & 0
\end{tikzcd}
$$

(α is unique, by the exactness of the top row, i.e., the injectivity of μ_1, but we don't really need this). Applying the commutativity of the diagram above (or, equivalently applying ∂_1 to both sides of (32) and using commutativity) we have

(33)
$$
\begin{tikzcd}
 & \alpha \arrow[r, mapsto] \arrow[ddd, dashed, bend right] & (\sigma_1,\sigma_2)-(\sigma_1',\sigma_2') \arrow[ddd, bend left] & & \\
0 \arrow[r] & \mathcal{C}_1(G_1\cap G_2) \arrow[r, "\mu_1"] \arrow[d, "\partial_1"'] & \mathcal{C}_1(G_1)\oplus\mathcal{C}_1(G_2) \arrow[r, "\nu_1"] \arrow[d, "\partial_1\oplus\partial_1"'] & \mathcal{C}_1(G) \arrow[r] \arrow[d, "\partial_1"'] & 0 \\
0 \arrow[r] & \mathcal{C}_0(G_1\cap G_2) \arrow[r, "\mu_0"] & \mathcal{C}_0(G_1)\oplus\mathcal{C}_0(G_2) \arrow[r, "\nu_0"] & \mathcal{C}_0(G) \arrow[r] & 0 \\
 & \partial_1\alpha \arrow[r, mapsto, dashed] & (\partial_1\sigma_1,\partial_1\sigma_2)-(\partial_1\sigma_1',\partial_1\sigma_2') & &
\end{tikzcd}
$$

and therefore

$$\mu_0(\partial_1\alpha) = (\partial_1\sigma_1, \partial_1\sigma_2) - (\partial_1\sigma_1', \partial_1\sigma_2').$$

In view of (31), we have

$$\mu_0(\partial_1\alpha) = \mu_0(\tau - \tau')$$

(this can also be obtained by comparing the bottom left corners of diagrams (33) and (30)). Since μ_0 is injective we have

$$\tau - \tau' = \partial_1\alpha.$$

Hence $\tau - \tau' \in \mathcal{C}_0(G_1 \cap G_2)$ equals $\partial_1\alpha$, and so $\tau - \tau'$ is in the image of ∂_1, and hence they lie in the same class (or coset) in the quotient space

$$H_0(G_1 \cap G_2) \stackrel{\text{def}}{=} \mathcal{C}_0(G_1 \cap G_2) \,/\, \partial_1\bigl(\mathcal{C}_1(G_1 \cap C_2)\bigr).$$

(6) Hence the procedure above takes $\beta \in H_1(G)$ to a $\tau \in \mathcal{C}_0(G_1 \cap G_2)$, and the class of τ in $H_0(G_1 \cap G_2)$ is independent of the choices made.

5.10. **Step Five of the Proof of Theorem 5.1.** Finally, another set of "diagram chasing" arguments show that the resulting sequence in Theorem 5.1 is exact.

For example, let's show that the part of the diagram:

(34)
$$H_1(G_1) \oplus H_1(G_2) \xrightarrow{\text{“}\nu_1\text{”}} H_1(G) \xrightarrow{\delta} H_0(G_1 \cap G_2)$$

is exact (in the position of $H_1(G)$. So we have to show that

$$\ker(\delta) = \mathrm{Image}(\text{“}\nu_1\text{”}).$$

First, let's show that $\ker(\delta)$ is contained in Image("ν_1"): so say that $\beta \in \ker(\delta)$: hence the $\tau = \delta(\beta)$ built in the last subsection equals 0; then we have:

$$\begin{tikzcd}
 & & (\sigma_1,\sigma_2) \arrow[rr, mapsto] \arrow[dddd, bend left, mapsto] & & \beta \arrow[dddd, bend left, mapsto] & \\
0 \arrow[r] & \mathcal{C}_1(G_1\cap G_2) \arrow[rr, "\mu_1"] \arrow[dd, "\partial_1"'] & & \mathcal{C}_1(G_1)\oplus\mathcal{C}_1(G_2) \arrow[r, "\nu_1"] \arrow[dd, "\partial_1\oplus\partial_1"'] & \mathcal{C}_1(G) \arrow[r] \arrow[dd, "\partial_1"'] & 0 \\
 & & & & & \\
0 \arrow[r] & \mathcal{C}_0(G_1\cap G_2) \arrow[rr, "\mu_0"] & & \mathcal{C}_0(G_1)\oplus\mathcal{C}_0(G_2) \arrow[r, "\nu_0"] & \mathcal{C}_0(G) \arrow[r] & 0 \\
 & \tau=0 \arrow[r, mapsto] & 0=\mu_0\tau=(\partial_1\sigma_1,\partial_1\sigma_2) \arrow[rr, mapsto] & & 0 &
\end{tikzcd}$$

and hence $(\partial_1\sigma_1, \partial\sigma_2) = 0$; it follows that (σ_1, σ_2) is in the kernel of the map

$$\mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2) \to \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2),$$

and so $(\sigma_1, \sigma_2) \in H_1(G_1) \oplus H_1(G_2)$, and under "$\nu_1$" (σ_1, σ_2) maps to β. Hence $\beta \in$ Image("ν_1").

Conversely, if $(\sigma_1, \sigma_2) \in H_1(G_1) \oplus H_1(G_2)$ and $\nu_1(\sigma_1, \sigma_2) = \beta$, so β is in the image of "ν_1", then

$$(\partial_1\sigma_1, \partial_1\sigma_2) = 0,$$

and so the unique $\tau \in \mathcal{C}_0(G_1 \cap G_2)$ with $\mu_0\tau = 0$

$$\begin{tikzcd}
 & & (\sigma_1,\sigma_2) \arrow[rr, mapsto] \arrow[dddd, bend left, mapsto] & & \beta \arrow[dddd, bend left, mapsto] & \\
0 \arrow[r] & \mathcal{C}_1(G_1\cap G_2) \arrow[rr, "\mu_1"] \arrow[dd, "\partial_1"'] & & \mathcal{C}_1(G_1)\oplus\mathcal{C}_1(G_2) \arrow[r, "\nu_1"] \arrow[dd, "\partial_1\oplus\partial_1"'] & \mathcal{C}_1(G) \arrow[r] \arrow[dd, "\partial_1"'] & 0 \\
 & & & & & \\
0 \arrow[r] & \mathcal{C}_0(G_1\cap G_2) \arrow[rr, "\mu_0"] & & \mathcal{C}_0(G_1)\oplus\mathcal{C}_0(G_2) \arrow[r, "\nu_0"] & \mathcal{C}_0(G) \arrow[r] & 0 \\
 & \tau \arrow[r, mapsto, dashed] & 0=(\partial_1\sigma_1,\partial_1\sigma_2) \arrow[rr, mapsto] & & 0 &
\end{tikzcd}$$

is necessarily $\tau = 0$.

Hence (34) is exact there.

We leave it to the reader as a sequence of EXERCISES to show that the rest of the sequence in Theorem 5.1 is exact.

5.11. **Examples of Theorem 5.1.**

Example 5.9. Consider Example 5.2. Let β be the cycle in $G = G_1 \cup G_2$ given by the closed walk (A, A', B, A'', A), and its corresponding 1-form

$$\beta = (A, A', B, A'', A)_{1-\text{form}} = [A, A'] + [A', B] + [B, A''] + [A'', A].$$

(DRAW PICTURE) Hence $\partial_1\beta = 0$, so $\beta \in H_1(G)$. Let us compute $\delta\beta$: first, need to find (σ_1, σ_2) with

$$\beta = \nu_1(\sigma_1, \sigma_2) = \sigma_1 - \sigma_2.$$

Given that A' is a vertex of only G_1, and A'' only of G_2, here we have no choice but to take:

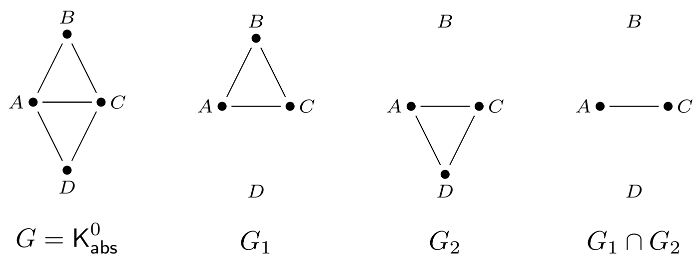
$$\sigma_1 = [A, A'] + [A', B], \quad \sigma_2 = -\big([B, A''] + [A'', A]\big).$$

We therefore easily compute

$$(\partial_1\sigma_1, \partial_1\sigma_2) = ([B] - [A], [B] - [A]),$$

and hence the unique τ with $\mu_0\tau = \beta$ is $\tau = [B] - [A] \in \mathcal{C}_0(G_1 \cap G_2)$, and this gives a class in $H_0(G_1 \cap G_2)$. Since $G_1 \cap G_2$ are isolated points, we have $\mathcal{C}_1(G_1 \cap G_2) = 0$ and hence $H_0(G_1 \cap G_2) = \mathcal{C}_0(G_1 \cap G_2)$; hence $\tau = [B] - [A]$ is a non-zero class in $H_0(G_1 \cap G_2) = \mathcal{C}_0(G_1 \cap G_2)$.

Example 5.10. Consider the graph $G = \mathsf{K}^0_{\mathsf{abs}}$ of Example 4.9: In other words, $G = \mathsf{K}^0_{\mathsf{abs}}$ is a graph, namely the complete graph on vertex set A, B, C, D with the edge $\{B, D\}$ omitted:



Let β be the 1-form corresponding to the closed walk (A, B, C, D, A), i.e.,

$$\beta = [A, B] + [B, C] + [C, D] + [D, A],$$

and hence $\partial_1\beta = 0$. Let's compute $\tau = \delta\beta$: we can write

$$\tau = \sigma_1 - \sigma_2, \quad \text{where} \quad \sigma_1 = [A, B] + [B, C], \quad \sigma_2 = -\bigl([C, D] + [D, A]\bigr);$$

but since G_1 and G_2 both contain the edge $\{A, C\}$, we could also write

$$\tau = \sigma_1' - \sigma_2', \quad \text{where} \quad \sigma_1' = [A, B] + [B, C] + 2025[A, C] \quad \sigma_2' = -\bigl([C, D] + [D, A]\bigr) + 2025[A, C]$$

(the 2025 is arbitrary). We easily compute

$$(\partial_1\sigma_1, \partial_1\sigma_2) = ([C] - [A], [C] - [A])$$

and hence $\tau = [C] - [A]$. However,

$$(\partial_1\sigma_1', \partial_1\sigma_2') = (2026[C] - 2026[A], 2026[C] - 2026[A])$$

It follows that $\tau' = 2026[C] - 2026[A]$. Notice that both τ, τ' are equal to 0 in $H_0(G_1 \cap G_2)$, since the image of ∂_1 in $G_1 \cap G_2$ is spanned by $\partial_1([A, C]) = [C] - [A]$.

5.12. **The General Mayer-Vietoris Sequence.** If K is an abstract simplicial complex of dimension d that is the union of two subcomplexes $\mathsf{K}_1, \mathsf{K}_2$, then we

similarly get a commutative diagram of short exact sequences (the horizontal rows):
(35)

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \mathcal{C}_{i+1}(\mathsf{K}_1 \cap \mathsf{K}_2) \arrow[r, "\mu_{i+1}"] \arrow[d, "\partial_{i+1}"'] & \mathcal{C}_{i+1}(\mathsf{K}_1) \oplus \mathcal{C}_1(\mathsf{K}_2) \arrow[r, "\nu_{i+1}"] \arrow[d, "\partial_{i+1} \oplus \partial_{i+1}"'] & \mathcal{C}_{i+1}(\mathsf{K}) \arrow[r] \arrow[d, "\partial_{i+1}"'] & 0 \\
0 \arrow[r] & \mathcal{C}_{i}(\mathsf{K}_1 \cap \mathsf{K}_2) \arrow[r, "\mu_{i}"] \arrow[d, "\partial_{i}"'] & \mathcal{C}_{i}(\mathsf{K}_1) \oplus \mathcal{C}_1(\mathsf{K}_2) \arrow[r, "\nu_{i}"] \arrow[d, "\partial_{i} \oplus \partial_{i}"'] & \mathcal{C}_{i}(\mathsf{K}) \arrow[r] \arrow[d, "\partial_{i}"'] & 0 \\
0 \arrow[r] & \mathcal{C}_{i-1}(\mathsf{K}_1 \cap \mathsf{K}_2) \arrow[r, "\mu_{i-1}"] \arrow[d] & \mathcal{C}_{i-1}(\mathsf{K}_1) \oplus \mathcal{C}_{i-1}(\mathsf{K}_2) \arrow[r, "\nu_{i-1}"] \arrow[d] & \mathcal{C}_{i-1}(\mathsf{K}) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

We similarly get the following theorem.

Theorem 5.11. *Let K be an abstract simiplicial complex of dimension d, and let K_1, K_2 be two subcomplexes of K whose union is K. Then there is a long exact sequence:*

$$
\begin{tikzcd}
0 \arrow[r] & H_d(G_1 \cap G_2) \arrow[r, "``\mu_d\text{''}"] & H_d(G_1) \oplus H_d(G_2) \arrow[r, "``\nu_d\text{''}"] & H_d(G) \arrow[dll, "\delta_d"'] & \\
 & H_{d-1}(G_1 \cap G_2) \arrow[r, "``\mu_{d-1}\text{''}"] & H_{d-1}(G_1) \oplus H_{d-1}(G_2) \arrow[r, "``\nu_{d-1}\text{''}"] & H_{d-1}(G) \arrow[dll, "\delta_{d-1}"'] & \\
 & \cdots & & \cdots \arrow[dll, "\delta_1"'] & \\
 & H_0(G_1 \cap G_2) \arrow[r, "``\mu_0\text{''}"] & H_0(G_1) \oplus H_{d-1}(G_2) \arrow[r, "``\nu_0\text{''}"] & H_0(G) \arrow[r] & 0
\end{tikzcd}
\tag{36}
$$

To prove the above theorem, it is more convenient (notationally) to prove the following general lemma.

Lemma 5.12. *Consider a commutative diagram of vector spaces:*

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{i+1} \arrow[r, "\mu_{i+1}"] \arrow[d, "d^A_{i+1}"'] & B_{i+1} \arrow[r, "\nu_{i+1}"] \arrow[d, "d^B_{i+1}"'] & C_{i+1} \arrow[r] \arrow[d, "d^C_{i+1}"'] & 0 \\
0 \arrow[r] & A_i \arrow[r, "\mu_i"] \arrow[d, "d^A_i"'] & B_i \arrow[r, "\nu_i"] \arrow[d, "d^B_i"'] & C_i \arrow[r] \arrow[d, "d^C_i"'] & 0 \\
0 \arrow[r] & A_{i-1} \arrow[r, "\mu_{i-1}"] \arrow[d] & B_{i-1} \arrow[r, "\nu_{i-1}"] \arrow[d] & C_{i-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
\tag{37}
$$

where the rows are short exact sequences, and the columns are chains. Then there is a ("long") exact sequence:

$$
\begin{tikzcd}
 & & \cdots \arrow[dll, "\delta_{i+2}"'] \\
H_{i+1}(\mathbf{A},\mathbf{d}^A) \arrow[r, "``\mu_{i+1}\text{''}"] & H_{i+1}(\mathbf{B},\mathbf{d}^B) \arrow[r, "``\nu_{i+1}\text{''}"] & H_{i+1}(\mathbf{C},\mathbf{d}^C) \arrow[dll, "\delta_{i+1}"'] \\
H_i(\mathbf{A},\mathbf{d}^A) \arrow[r, "``\mu_i\text{''}"] & H_i(\mathbf{B},\mathbf{d}^B) \arrow[r, "``\nu_i\text{''}"] & H_i(\mathbf{C},\mathbf{d}^C) \arrow[dll, "\delta_i"'] \\
\cdots & &
\end{tikzcd}
\tag{38}
$$

(where "μ_i" is the natural map on quotient spaces induced from μ_i, and similarly for other maps in quotation marks, and the δ_j are the "delta maps" obtained by the analogous procedure as used to prove Theorem 5.1).

The exact sequence (38) is called the "long exact sequence" associated to the commutative diagram of short exact sequences (37). The proof of Lemma 5.12 is a similar set of "diagram chasing" arguments.

EXERCISES: Prove the above lemma. Most likely we will assign only part of the proof.

Proof of Theorem 5.11. By the definition of $\mathcal{C}_i(\mathsf{K}_{\mathsf{abs}})$, we easily verify that (35) is a commutative diagram whose rows are short exact sequences, where we set

$$\mu_j(\tau) = (\tau, \tau), \quad \nu_j(\sigma_1, \sigma_2) = \sigma_1 - \sigma_2.$$

Now we apply Lemma 5.12. $\square$

6. Cones

We have already introduced the notion of the cone $\mathrm{Cone}_P(\mathsf{K})$ about P of an abstract simplicial complex K. We also proved that the Betti numbers of any cone are $\beta_0 = 1$ (i.e., any cone is connected), and $\beta_i = 0$ for $i > 0$.

On Feb 14, 2025 we will review the definition of a cone, and recall that its Betti numbers are as above. We will likely skip the rest of this section, perhaps returning to it later.

In topology, there are many ways of getting new topological spaces from old ones, including taking the cartesian products, quotients, and joins; however, of these three operations, only the join takes simplicial complexes and directly produces a simplicial complexes. The join can also has a simple analog for abstract simplicial complexes. The simplest non-trivial case of a join is called a *cone*, which we now define.

Let K be an abstract simplicial complex, and let P not lie in $V(\mathsf{K})$. The *cone of* K *at* P, which we will denote $\mathrm{Cone}_P(K)$ refers to the set

$$= \mathsf{K} \cup \{A \cup P \mid A \in \mathsf{K}\},$$

which we easily see is an abstract simplicial complex.

DRAW PICTURE

EXERCISE: Let K be a simplicial complex in $\mathbb{R}^N$ whose associated simplicial complex is K. Then, identifying $\mathbb{R}^{N+1}$ with $\mathbb{R}^N \times \mathbb{R}$, the set

$$\{(\alpha_0 \mathbf{x}, \alpha_1) \in \mathbb{R}^{N+1} \mid \mathbf{x} \in |K|, 0 \le \alpha \le 1\}$$

is the geometric realization of a simplicial complex whose abstract simplicial set is $\mathrm{Cone}_P(\mathsf{K})$ where $P = (\mathbf{0}, 1)$.

The graph of $\mathrm{Cone}_P(K)$ contains an edge joining P and any vertex of K; hence $\mathrm{Cone}_P(K)$ is connected, and $H_0^{\mathrm{simp}}(\mathrm{Cone}_P(K)) \simeq \mathbb{R}$.

Theorem 6.1. *Let* $\mathsf{L} = \mathrm{Cone}_P(\mathsf{K})$ *be a cone of a complex* K*. Then* $H_i(\mathsf{L}) = 0$ *for* $i \ge 1$*.*

The case where K is a graph, even a graph with no edges (i.e., the graph consists of some number of isolated vertices) illustrates the main ideas in the proof we give.

DRAW THIS AND EXPLAIN WHY THE THEOREM IS TRUE IN THIS CASE.

We will give three proofs of this Theorem 6.1. The third proof is the quickest, but uses the fact that a simplicial complex associated to a cone is *contractible*. The first two proofs can be given on the level of combinatorics and linear algebra, and both are instructive.

First proof of Theorem 6.1. Let $\alpha \in \mathcal{C}_k(\mathsf{L})$ satisfy $\partial_k \tau = 0$; we have to show that there is a $\sigma \in \mathcal{C}_{k+1}$ for which $\partial_{k+1}\sigma = \tau$. We claim that it suffices to do this when τ is of the form

$$\tau = \sum_{i=1}^{r} \alpha_i [P, v_{i1}, \ldots, v_{ik}] : \tag{39}$$

indeed, if τ contains terms of the form $\alpha[u_0,\ldots,u_k]$ where each u_i is distinct from P, then $[P,u_0,\ldots,u_k]$ is a $(k+1)$-form, and

$$\partial_{k+1}[P,u_0,\ldots,u_k]=[u_0,\ldots,u_k]+\sum_{i=0}^{k}[P,u_0,\ldots,\widehat{u}_i,\ldots u_k](-1)^{i+1}.$$

Hence, modulo the image of ∂_{k+1}, any τ is of the form (39); hence it suffices to show that such a τ is in the image of ∂_{k+1}.

Since $\partial_k\tau=0$, it follows that all the sum of all $(k-1)$-forms that don't involve P of $\partial_k\tau$ must equal zero. But these terms of $\partial_k\tau$ equal:

$$\sum_{i=0}^{k}\alpha_i[v_{i1},\ldots,v_{ik}](-1)^{i+1}.$$

Hence this sum is zero, and hence so is (39). $\square$

Second proof of Theorem 6.1. Let $\mathcal{C}_i=\mathcal{C}_i(\mathsf{L})$. We will build maps $K_i\colon\mathcal{C}_i\to\mathcal{C}_{i+1}$ for $i\in\mathbb{Z}_{\geq 0}$ such that for all $\tau\in\mathcal{C}_i$ we have

$$\forall i\geq 1,\tau\in\mathcal{C}_i,\quad \tau=\bigl(\partial_{i+1}K_i+K_{i-1}\partial_i\bigr)\tau \tag{40}$$

We visualize the maps $\{K_i\}$ in the diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_3"] & \mathcal{C}_2 \arrow[r, "\partial_2"] \arrow[d] \arrow[dl, "K_2"'] & \mathcal{C}_1 \arrow[r, "\partial_1"] \arrow[d] \arrow[dl, "K_1"'] & \mathcal{C}_0 \arrow[r, "\partial_0"] \arrow[d] \arrow[dl, "K_0"'] & 0 \\
\cdots \arrow[r, "\partial_3"] & \mathcal{C}_2 \arrow[r, "\partial_2"] & \mathcal{C}_1 \arrow[r, "\partial_1"] & \mathcal{C}_0 \arrow[r, "\partial_0"] & 0
\end{tikzcd} \tag{41}$$

notice that the above diagram is NOT a commutative diagram, but is the usual notion of a *homotopy of chain maps*; see below. To build the maps $K_i\colon\mathcal{C}_i\to\mathcal{C}_{i+1}$, we set K_i to be the unique linear map such that

$$K_i\bigl([P,u_0,\ldots,u_{i-1}]\bigr)=0$$

assuming $u_0,\ldots,u_{i-1},P$ are distinct, and

$$K_i\bigl([u_0,\ldots,u_i]\bigr)=[P,u_0,\ldots,u_i]$$

assuming $P,u_0,\ldots,u_i$ are distinct. Now we check:

$$\bigl(\partial_{i+1}K_i+K_{i-1}\partial_i\bigr)[P,u_0,\ldots,u_{i-1}]=K_{i-1}\partial_i[P,u_0,\ldots,u_{i-1}],$$

and since K_{i-1} is zero on any term containing P, the above equals

$$=K_{i-1}[u_0,\ldots,u_{i-1}]=[P,u_0,\ldots,u_{i-1}],$$

which verifies (40) for $\tau=[u_0,\ldots,u_{i-1},P]$. If $u_0,\ldots,u_i$ are distinct, we have

$$\bigl(\partial_{i+1}K_i+K_{i-1}\partial_i\bigr)[u_0,\ldots,u_i]=\partial_{i+1}[P,u_0,\ldots,u_i]+K_{i-1}\sum_{j=0}^{i}[u_0,\ldots,\widehat{u}_j,\ldots u_i](-1)^j$$

$$=[u_0,\ldots,u_i]+\sum_{j=0}^{i}[P,u_0,\ldots,\widehat{u}_j,\ldots u_i](-1)^{j+1}+\sum_{j=0}^{i}[P,u_0,\ldots,\widehat{u}_j,\ldots u_i](-1)^j$$

$$=[u_0,\ldots,u_i].$$

Now say that $i \geq 1$ and $\tau \in \ker(\partial_i)$. Then $\partial_i \tau = 0$, and hence (40) implies that

$$\tau = \partial_{i+1}(K_i \tau),$$

and therefore $\tau \in \mathrm{Image}(\partial_{i+1})$. Hence $\ker(\partial_i) = \mathrm{Image}(\partial_{i+1})$, and hence $H_i(\mathsf{L}) = 0$. $\square$

Remark 6.2. There is a simpler way to describe the above maps K_i: first, if $u_0, \ldots, u_i \in V(\mathsf{K})$ are vertices of a simplicial complex, then it makes sense to interpret

$$[u_0, u_0, u_1, u_2, \ldots, u_i] \in \mathcal{C}_{i+1}(\mathsf{K})$$

as zero, since exchanging the first two u_0 we should have

$$[u_0, u_0, u_1, u_2, \ldots, u_i] = -[u_0, u_0, u_1, u_2, \ldots, u_i],$$

and since we are working over $\mathbb{R}$, we should have

$$[u_0, u_0, u_1, u_2, \ldots, u_i] = 0.$$

With these conventions, K_i is the unique linear map taking $[u_0, \ldots, u_i]$ to $[P, u_0, \ldots, u_i]$ (which equals zero if any of $u_0, \ldots, u_i$ equal P).

Remark 6.3. Generally speaking, if f, g are maps $(\mathcal{C}, \boldsymbol{\partial})$ to $(\mathcal{C}', \boldsymbol{\partial}')$, then we say that *$f$ and g are homotopic* if there exist $K_i \colon \mathcal{C}_i \to \mathcal{C}'_{i+1}$ for all i such that $f - g = \partial'_i K_i + K_{i-1}\partial_i$. This same argument shows that f, g yield the same maps on homology, i.e., $f_* = g_*$ as maps $H_i(\mathcal{C}) \to H_i(\mathcal{C}')$ for each i. The second proof of Theorem 6.1 takes $(\mathcal{C}', \boldsymbol{\partial}') = (\mathcal{C}, \boldsymbol{\partial})$, $f = \mathrm{id}_{\mathcal{C}}$, and $g \colon \mathcal{C} \to \mathcal{C}$ is given by $g \colon \mathcal{C}_i \to \mathcal{C}_i$ is 0 for $i \geq 1$, and $g \colon \mathcal{C}_0 \to \mathcal{C}_0$ is given by $g(v) = P$ for all $v \in V(\mathsf{L})$.

Remark 6.4. We may wait until later for this remark. Let L, L' be two abstract simplicial complexes, with vertex sets V, V' respectively. A *morphism* $\phi \colon \mathsf{L} \to \mathsf{L}'$ is a map $\phi \colon V \to V'$ such that $A \in \mathsf{L}$ implies that $\phi(A) \in \mathsf{L}'$, where we extend ϕ to be defined on all subsets of V in the evident sense, i.e., $\phi(A) = \{\phi(a) | a \in A'\}$. Such a morphism ϕ gives rise to a map $\phi_\# \colon \mathcal{C}_i(\mathsf{L}) \to \mathcal{C}_i(\mathsf{L}')$ given by $\phi_\#[v_0, \ldots, v_i] \mapsto [\phi(v_0), \ldots, \phi(v_i)]$, with the understanding that $[u_0, \ldots, u_i] = 0$ if $u_0, \ldots, u_i$ are not all distinct. [This is really "forced upon us," at least when working over $\mathbb{R}$, since if we exchange two elements in the sequence $[u_0, \ldots, u_i]$ we should get the same sequence with a $-$ sign.] If $\mathsf{L} = \mathrm{Cone}_P(\mathsf{K})$ and $\mathsf{L}' = \{\emptyset, \{P'\}\}$ for some P', then L' has P' as its only vertex, and hence there is a single morphism $\phi \mathsf{L} \to \mathsf{L}'$ (taking P plus the vertices of K to P'). Let $\psi \colon \mathsf{L}' \to \mathsf{L}$ be the morphism taking P' to P. Then clearly $\phi_\#\psi_\#$ is the identity morphism; we claim that $g = \psi_\#\phi_\#$ is precisely the map g described in the previous remark. Hence, although $g = \psi_\#\phi_\#$ is not the identity map (unlike $\phi_\#\psi_\#$), g is *homotopic* to the identity map. Now explain the notion of two chains being *chain homotopic*.

Remark 6.5. The third proof of Theorem 6.1 is based on the fact that the geometric realization of L is *contractible*, i.e., homotopic to the point P. However, the chain homotopy in the previous remark is really a reflection of the fact that L is contractible to P.

Remark 6.6. Notice that (40) does not hold for $i = 0$. In this case, the RHS of (40) is $K_0\partial_1$ which for $u \neq P$ takes $[u]$ to $[u] - [P]$ and also takes $[P]$ to $0 = [P] - [P]$. We will see this a yielding a "chain homotopy" between $\mathcal{C}(\mathsf{L}_{\mathsf{abs}})$ and $\mathcal{C}(\{\emptyset, P\})$. This is a reflection of the previous remark.

7. Suspensions, and the Suspension of a Simplicial Complex in $\mathbb{R}$

Definition 7.1. Let K be a simplicial complex on a vertex set V, and $P_1, P_2 \notin V$ be distinct. We define the *suspension of* K *at* P_1, P_2 to be

$$\mathrm{Susp}(\mathsf{K}) = \mathrm{Suspension}_{P_1,P_2}(\mathsf{K}) \stackrel{\mathrm{def}}{=} \mathrm{Cone}_{P_1}(\mathsf{K}) \cup \mathrm{Cone}_{P_2}(\mathsf{K}).$$

Note also that

$$\mathrm{Cone}_{P_1}(\mathsf{K}) \cap \mathrm{Cone}_{P_2}(\mathsf{K}) = \mathsf{K}.$$

Clearly P_1, P_2 are connected to each vertex of K in $\mathrm{Susp}(\mathsf{K}) = \mathrm{Susp}_{P_1,P_2}(\mathsf{K})$, and hence $H_0(\mathrm{Susp}(\mathsf{K})) \simeq \mathbb{R}$ and $\beta_0(\mathrm{Susp}(\mathsf{K})) = 1$. The other Betti numbers can be computed from the Mayer-Vietoris sequence (Subsection 5.12), setting

$$\mathsf{L}_1 = \mathrm{Cone}_{P_1}(\mathsf{K}), \quad \mathsf{L}_2 = \mathrm{Cone}_{P_2}(\mathsf{K}),$$

which gives

$$\mathsf{L}_1 \cap \mathsf{L}_2 = \mathsf{K}, \quad \mathsf{L}_1 \cup \mathsf{L}_2 = \mathsf{L} = \mathrm{Susp}(\mathsf{K}).$$

We then have

$$\forall i \geq 1, \quad H_i(\mathsf{L}_j) = 0.$$

And we easily see that:

$$\beta_1(\mathrm{Susp}(\mathsf{K})) = \beta_0(\mathsf{K}) - 1$$

(assuming the vertex set of K is non-empty), and that

$$\forall i \geq 1, \quad \beta_{i+1}\bigl(\mathrm{Susp}(\mathsf{K})\bigr) \simeq \beta_i(\mathsf{K}).$$

DRAW SOME PICTURES

One feature of the suspension is that the suspension of the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, as a *topological space*, is the sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$. In Section 9 we discuss *topological spaces* and make this precise.

Remark 7.2. If K and K' are two abstract simplicial complexes with disjoint vertex sets, we define their *join* to be

$$\mathsf{K} * \mathsf{K}' = \{A \cup A' \mid A \in \mathsf{K},\ A' \in \mathsf{K}'\}.$$

We will see that there is a corresponding *join* operation of two simplicial complexes.

Remark 7.3. This may be a good time to talk about *reduced homology*, where one adds a $\mathcal{C}_{-1}(\mathsf{K}) = \mathbb{R}$ for $\emptyset \in \mathsf{K}$, which one considers to be a face of dimension -1, along with the map $\partial_0 \colon \mathcal{C}_0(\mathsf{K}) \to \mathcal{C}_{-1}(\mathsf{K})$ that for each $v \in V$, $\partial_0(v) = \emptyset$. Said otherwise, $\partial(\sum_i \alpha_i v_i) = \sum_i \alpha_i \in \mathbb{R}$. This only changes $H_0^{\mathrm{simp}}(\mathsf{K})$, and the result, denoted $\tilde{H}_0^{\mathrm{simp}}(\mathsf{K})$, has dimension one smaller than $H_0^{\mathrm{simp}}(\mathsf{K})$; all other reduced homology groups, $\tilde{H}_i^{\mathrm{simp}}(\mathsf{K})$ equal $H_i^{\mathrm{simp}}(\mathsf{K})$. In terms of reduced homology we have

$$\forall i \geq 0, \quad \tilde{\beta}_{i+1}\bigl(\mathrm{Susp}(\mathsf{K})\bigr) \simeq \tilde{\beta}_i(\mathsf{K}),$$

(not just for $i \geq 1$) where $\tilde{\beta}_i$ is the *reduced* i*-th Betti number*, $\tilde{\beta}_i(\mathsf{K}) = \dim(\tilde{H}_i^{\mathrm{simp}}(\mathsf{K}))$. For reasons like this, the reduced Betti numbers are sometimes more convenient than regular Betti numbers.

Remark: As Rain Y pointed out in 2025, a number of the exercises in Section A would be easier with reduced homology.

Remark 7.4. We caution the reader that there is another notion of reduced Betti numbers, mainly for graphs, where $\beta_0^{\mathrm{red}}(G)$ is the number of disconnected components of G that are trees, and $\beta_1^{\mathrm{red}}(G)$ is $\beta_0^{\mathrm{red}}(G)-\chi(G)$, i.e., $\beta_1(G)$ minus the number of connected components of G that have at least one cycle. This notion of reduced Betti number behaves well under covering maps (lifts), and plays a prominent role in combinatorial group theory; see, e.g., [Fri15].

8. Discrete Hodge Theory

For a graph, we defined two Laplacians, the *edge Laplacian* $\partial_1^{\mathrm{T}}\partial_1$, and the vertex Laplacian $\partial_1\partial_1^{\mathrm{T}}$. Moreover, for a general simplicial complex, K, we defined the k-th Laplacian to be

$$\Delta_{\mathsf{K}}^k=\partial_{k-1}^{\mathrm{T}}\partial_k+\partial_k\partial_{k+1}^{\mathrm{T}}.$$

We now explain our interest in this construction.

Let $\mu\colon A\to B$ and $\nu\colon B\to C$ be maps of vector spaces, which we write more concisely as $A\xrightarrow{\mu}B\xrightarrow{\nu}C$. Say that $\nu\mu=0$, and hence $\mathrm{Image}(\mu)\subset\ker(\nu)$, and we can define

$$H\stackrel{\mathrm{def}}{=}\ker(\nu)/\mathrm{Image}(\mu).$$

Some examples are:

$$\mathbb{R}\xrightarrow{1}\mathbb{R}\xrightarrow{0}\mathbb{R},\quad \mathbb{R}\xrightarrow{0}\mathbb{R}\xrightarrow{1}\mathbb{R},\quad \mathbb{R}\xrightarrow{0}\mathbb{R}\xrightarrow{0}\mathbb{R}, \tag{42}$$

where 1 above is the identity map, and 0 is the zero map; the corresponding values of H are, respectively, $0,0,\mathbb{R}$. Then point is that for the purposes of determining H, the examples (42) are essentially the *only* examples, in the following sense.

Proposition 8.1. *Let $A\xrightarrow{\mu}B\xrightarrow{\nu}C$ be maps of vector spaces such that $\nu\mu=0$ (and therefore $\mathrm{Image}(\mu)\subset\ker(\nu)$). Then there are bases of A,B,C with respect to which μ,ν in this basis equal, in block form:*

$$\begin{bmatrix} I_m & 0 & 0 & 0\\ 0 & 0_p & 0 & 0\\ 0 & 0 & 0_n & 0\end{bmatrix},\quad \begin{bmatrix} 0_m & 0 & 0\\ 0 & 0_p & 0\\ 0 & 0 & I_n\\ 0 & 0 & 0\end{bmatrix}.$$

where $m+p+n=\dim(B)$, and the subscripts in the diagonal blocks gives the dimensions of these square matrices (e.g., I_m refers to the $m\times m$ identity matrix, and 0_n refers to the $n\times n$ zero matrix), and otherwise these block are matrices of compatible dimensions (and we don't care how many zero columns there are in the last block column of the first matrix and the second block column of the second matrix). Hence $\Delta\stackrel{\mathrm{def}}{=}\nu\nu^{\mathrm{T}}+\mu^{\mathrm{T}}\mu$ in block form, with respect to the basis for B equals

$$\begin{bmatrix} I_m & 0 & 0\\ 0 & 0_p & 0\\ 0 & 0 & I_n\end{bmatrix},$$

and

$$\dim(H)=\dim\bigl(\ker(\nu)/\mathrm{Image}(\mu)\bigr)=p,$$

and

$$\ker(\Delta)=\bigl(0^m\times\mathbb{R}^p\times 0^n\bigr)_{\text{the basis for }B}\simeq H.$$

Moreover, below we review what is meant by an *inner product* on a vector space; we will then show that if in Proposition 8.1 A, B, C are each endowed with an inner product, then, the bases in Proposition 8.1 can be chosen to be orthonormal bases.

Notice that in Proposition 8.1 the extra block column in the block form of μ has no effect on Image(μ), since it represents elements of A that are taken to 0. Similarly for the extra row in that of ν. Hence one can view the “interesting part” of the sequence in $A \xrightarrow{\mu} B \xrightarrow{\nu} C$ as being a sum of the examples in (42). Of course, one could further ignore the second and third columns in the block description of μ, since they don't contribute to the image of μ; however, keeping the square matrices makes it easier to understand the Laplacian and Hodge decomposition.

Corollary 8.2. *In Proposition 8.1, B equals the internal direct sum*

$$\mathrm{Image}(\mu) \oplus \mathcal{H} \oplus \mathrm{Image}(\nu^T), \tag{43}$$

where

$$\mathcal{H} \stackrel{def}{=} \ker(\Delta).$$

The the first two summands of (43) are taken to zero by ν, and the last two summands are taken to zero by μ^{T}. Moreover $H \simeq \mathcal{H}$, and more precisely each element of H has a single representative $b \in \mathcal{H}$, and for any other representative b' of the class of b in H we have

$$(b, b) \leq (b', b') \tag{44}$$

with equality iff $b' = b$, where (b, b) is the standard inner product $b^{\mathrm{T}} b$.

When we generalize Proposition 8.1 to inner product spaces, then (44) holds with respect to the given inner product on B.

TO BE CONTINUED...

Part 3. The Singular Homology of a Topological Space

9. Topological Spaces and Singular Homology

The Betti numbers of a graph, G, that is a cycle of some length, are $\beta_0 = \beta_1 = 1$, and do not depend on the length. In this section we will explain why this is true. In fact, we will develop a set of powerful tools that provide a lot of intuition regarding the simplicial homology groups, namely *singular homology* and its (most convenient) setting of *topological spaces*.

9.1. Overview of Singular Homology.

In this section we will define the *singular homology groups*, $H_i(X) = H_i^{\mathrm{sing}}(X)$ for any subset $X \subset \mathbb{R}^n$. We will list some fundamental results about the groups $H_i^{\mathrm{sing}}(X)$, (mostly without proof) such as:

(1) Let K be a simplicial complex in $\mathbb{R}^n$ (and recall that $|K| = |K|_{\mathrm{geom}}$ refers to the union of the elements of K, which is therefore a subset of $\mathbb{R}^n$). Then if K is the abstract simplicial complex associated to K, then

$$H_i^{\mathrm{sing}}(|K|) \simeq H_i^{\mathrm{simp}}(\mathsf{K}).$$

In this sense, the singular homology and simplicial homology agree.

(2) We say that $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are *homeomorphic* if there is a bijection $f \colon X \to Y$ such that f and f^{-1} are *continuous* (we will review the notion of continuity). If so, then
$$H_i^{\mathrm{sing}}(X) \simeq H_i^{\mathrm{sing}}(Y).$$
(This isomorphism holds even when X, Y are *homotopic*, which is a much more general condition.)

We easily see that if K is an abstract simplicial complex that is a graph that is a cycle of length k, then K is associated to a simplicial complex K in $\mathbb{R}^2$ such that $|K|$ is isomorphic to the circle:
$$\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2.$$
It follows that
$$H_i^{\mathrm{simp}}(\mathsf{K}) \simeq H_i^{\mathrm{sing}}(\mathbb{S}^1),$$
where K is a cycle of any length.

The second aspect of the singular homology groups $H_i^{\mathrm{sing}}(X)$ is that they are defined not only for $X \subset \mathbb{R}^n$, but in the much broader context when X is any *topological space*. In fact, to study singular homology, at times it is needlessly cumbersome — or even impossible — to find a subset of some $\mathbb{R}^m$ for some m that is (homeomorphic to a) topological space we want to use. One crucial example of this is a space $X/\sim$, obtained from an equivalence relation $\sim$ on a topological space, X, where: (1) $X/\sim$ is not generally homeomorphic to a subset of $\mathbb{R}^m$ for some m, and (2) even when $X/\sim$ is homeomorphic to some subset of $\mathbb{R}^m$ for some m (such as for toruses, Δ-complexes, etc.), it can be quite cumbersome to find such a subset $S \subset \mathbb{R}^m$ and/or to use such an $S \subset \mathbb{R}^m$ as a replacement of $X/\sim$. Topological spaces are a more versatile tool that will allow us to easily build many important spaces needed to understand homology groups. Moreover, topological spaces discard a lot of extraneous information, and gives the minimum structure we need to define continuous maps.

9.2. **Thinking "Geometrically".** Until now, we have only defined the *simplicial homology groups*, $H_i^{\mathrm{simp}}(\mathsf{K})$, where K is a finite, combinatorial structure. However, we originally defined a *simplicial complex*, K, as a set of simplicies in $\mathbb{R}^n$ (for some n) such that (1) if $X \in K$, then any face of X lies in K, and (2) if $X, X' \in K$, then $X \cap X'$ is a face of both X and of X'. We then defined
$$|K| = |K|_{\mathrm{geom}} = \bigcup_{X \in K} X$$
which is a subset of $\mathbb{R}^n$. If K is the abstract simplicial complex associated to K, and if K' is that of another simplicial complex K' in $\mathbb{R}^m$, then
$$\forall i, \quad H_i^{\mathrm{simp}}(\mathsf{K}) \simeq H_i^{\mathrm{simp}}(\mathsf{K}') \tag{45}$$
whenever $|K| \subset \mathbb{R}^n$ and $|K'| \subset \mathbb{R}^m$ are *homeomorphic*, meaning that there is a bijection $f \colon |K| \to |K'|$ such that f and f^{-1} are *continuous*. So we need to develop some intuition (and the formal definition) of *homeomorphic* sets.

[Later it will turn out that (45) holds provided that $|K|$ and $|K'|$ are of the same *homotopy type*; we'll need to also develop some intuition there...]

In class on Feb 24, 2025, we drew pictures of the following two examples.

Example 9.1. Let K be a cycle of length 6. Then K is the abstract complex associated to K which is the collection of vertices and line segments that comprise a regular hexagon in $\mathbb{R}^2$ with vertices on the unit circle. Hence $|K| \subset \mathbb{R}^2$ is just a hexagon (without its interior) inscribed in the unit circle, and polar coordinates sets up a *homeomorphism* $|K| \to \mathbb{S}^1$. Similarly if K is a cycle of any length.

Example 9.2. The following abstract simplicial complexes, K have isomorphic homology groups (their Betti numbers, $\beta_i(\mathsf{K}) = \dim(H_i(\mathsf{K}))$ are $\beta_0 = \beta_2 = 1$, and all other Betti numbers vanish:

(1) $\mathsf{K} = \mathrm{Power}(S) \setminus \{S\}$ where $|S| = 4$, and
(2) the suspension of any complex that is a (graph that is a) cycle of some length.

Examining some pictures (e.g., on the Feb 24, 2025 class board scans), we see that each of these K is the abstract simplicial complex of the simplicial complex K, where $|K|$ is homeomorphic to

$$\mathbb{S}^2 = \bigl\{(x_1, x_2, x_3) \in \mathbb{R}^3 \;\bigm|\; x_1^2 + x_2^2 + x_3^3 = 1\bigr\}.$$

Remark 9.3. There is a standard remark that a (solid) donut (doughnut) and a (solid) coffee cup are *homeomorphic*. The same is true of the two-dimensional surface that is the boundary of a donut and that of a coffee cup, and both surfaces are homeomorphic to a (two-dimensional) torus (which we will soon define). This fact lends itself to numerous quips involving topologists, coffee, and donuts.

Example 9.4. We've seen that if $\mathsf{K}_{\mathsf{abs}}$ is any abstract simplicial complex, and $\mathsf{L}_{\mathsf{abs}} = \mathrm{Cone}_P(\mathsf{K}_{\mathsf{abs}})$ is a cone, then $\mathsf{L}_{\mathsf{abs}}$ have the same Betti numbers as that of a single point. There is a "geometric" reason for this. Namely, we say that $X \subset \mathbb{R}^n$ is *contractible* if there is a function $f\colon X \times [0,1] \to X$ ($[0,1]$ is a unit interval, so $X \times [0,1] \subset \mathbb{R}^{n+1}$ such that for some $x_0 \in X$ we have

$$\forall x \in X, \quad f(x,0) = x, \quad f(x,1) = x_0.$$

On Feb 26, 2025, we drew some pictures of what this means.

To rigorously work with the above ideas, we need to review what we mean by a *continuous map*.

9.3. **Continuous Maps, Limits, and Relative Neighbourhoods.** The main point of this subsection is to motivate the definition of a *topological space*. A secondary goal is to point out the interesting approach of Armstrong's textbook, namely in Section 1.4 of [Arm83], where one first defines "neighbourhoods" in a wide sense (these "neighbourhoods" are not necessarily open sets), as a way to intuitively transition from continuous maps in the limit sense to topological spaces. Armstrong does something more extreme, namely to define a topological space as a set, X, plus, for each $x \in X$, a set of (wide-sense) neighbourhoods of x that satisfy certain properties; the wide-sense neighbourhoods then give rise to the topology (open sets) of X. Here we do something less extreme: we use wide-sense neighbourhoods as a way of motivating topological spaces.

Let us begin with the usual definition of continuous maps that one encounters in calculus.

Definition 9.5. Let $X \subset \mathbb{R}^n$ for some n, and let $f\colon X \to \mathbb{R}^m$ be a function. If $x_0 \in X$, we say that f is *continuous* at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0). \tag{46}$$

We say that f is *continuous* if it continuous at all $x_0 \in X$.

For this definition to make sense, we need to know what is meant by the limit in (46). Intuitively, this means that $f(x)$ can be made as close as we like to $f(x_0)$, by taking $x \in X$ sufficiently close to x_0. In other words, we define (46) to mean:

> For any $\epsilon > 0$ there is an $\delta > 0$ such that $|f(x) - f(x_0)| \le \epsilon$ provided that $x \in X$ and $|x - x_0| \le \delta$.

Often one replaces the $\le$ in the above with $<$; the two definitions are equivalent.

Notice that if in Definition 9.5, $X = [a, b] \subset \mathbb{R}$ with $a < b$, and so $f\colon [a, b] \to \mathbb{R}^m$, we understand that $f(x)$ is not defined for $x < a$. Hence (46) is a "one-sided" limit when $x_0 = a$; similarly when $x_0 = b$. So we emphasize that (46) always assumes that $x \in X$, i.e., the limit is taken over those $x \in X$ where $f(x)$ is defined. We next describe a situation where the phrase $x \in X$ is extraneous, by defining "neighbourhoods."

9.3.1. *Balls and Neighbourhoods.*

Definition 9.6. Let $x \in \mathbb{R}^n$ for some n. For any real $\rho > 0$, we define the *closed ball* (respectively, *open ball*) *of radius* ρ *about* x to be the respective sets

$$B_\rho^{\mathrm{clos}}(x) = \bigl\{x' \in \mathbb{R}^n \ \bigm|\ |x - x'| \le \rho\bigr\}, \quad B_\rho^{\mathrm{open}}(x) = \bigl\{x' \in \mathbb{R}^n \ \bigm|\ |x - x'| < \rho\bigr\}.$$

We say that a set $N \subset \mathbb{R}^n$ is a *neigbourhood of* x if for some $\delta > 0$ we have that $B_\rho^{\mathrm{clos}}(x) \subset N$.

Notice that in the definition of neighbourhood above, it would be equivalent to require that $B_\rho^{\mathrm{open}}(x) \subset N$; this is because for any $\rho < \rho'$ we have

$$B_\rho^{\mathrm{open}}(x) \subset B_\rho^{\mathrm{clos}}(x) \subset B_{\rho'}^{\mathrm{open}}(x).$$

Returning to (46), if $x_0 \in X$ and X is a neighbourhood of x, then $f(x)$ in (46) is defined for all $x \in B_\rho^{\mathrm{clos}}(x_0)$ for some $\rho > 0$. Then the limit (46) slightly simplifies to:

> For any $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ provided that $|x - x_0| < \delta$;

in other words, we can drop the condition $x \in X$, namely by replacing δ by $\min(\delta, \rho)$ if need be.

9.3.2. *Neighbourhood Definition of Continuity.* Now we get to the following simpler definition of continuity.

Theorem 9.7. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function for some n, m. If $x_0 \in \mathbb{R}^n$, then f is continuous iff for any neighbourhood, N, of $f(x_0)$,*

$$f^{-1}(N) = \bigl\{x \in X \ \bigm|\ f(x) = N\bigr\}$$

is a neighbourhood of x_0.

The proof is an EXERCISE, but there is the point: the condition $|f(x)-f(x_0)| \le \delta$ can be expressed as $f(x) \in B_\delta^{\rm clos}(f(x_0))$, and $N = B_\delta^{\rm clos}(f(x_0))$ is certainly a neighbourhood of $f(x_0)$. If $f^{-1}(N)$ contains a neighbourhood of x, then it contains all x such that $|x - x_0| \le \epsilon$ for some $\epsilon > 0$.

It is not hard to generalize the above theorem as follows.

Definition 9.8. If $X \subset \mathbb{R}^n$ is a subset, for some n, and $x \in X$, then a *relative neighbourhood of x in X (in viewing X as a subset of $\mathbb{R}^n$)* is any set of the form $M \cap X$ where M is a neighbourhood of x.

Theorem 9.9. *Let $X \subset \mathbb{R}^n$ for some n, and let $f\colon X \to \mathbb{R}^m$ be a function. If $x_0 \in X$, then f is continuous at x_0 iff for each neighbourhood, N, of $f(x_0)$, $f^{-1}(N)$ is a neighbourhood of x in X.*

The proof is an EXERCISE.

Theorem 9.9 is terrific news: it tells us that continuity can be defined as soon as you know that is meant by a "neighbourhood" of the domain and the codomain (or range).

Now we get to an even simpler definition of continuity.

9.3.3. *Open Subset Definition of Continuity.*

Definition 9.10. A subset $N \subset \mathbb{R}^n$ is *open* if it is a neighbourhood of all its elements, i.e., for each $x \in N$, there is a $\rho > 0$ such that $B_\rho^{\rm clos}(x) \subset N$.

For example, an open interval $(a, b) \subset \mathbb{R}$ is an open subset of $\mathbb{R}$.

The theorems below are exercises; if you have never seen them, you should make sure that you understand them.

Theorem 9.11. *If $f\colon \mathbb{R}^n \to \mathbb{R}^m$, then f is continuous iff for every open subset, $U \subset \mathbb{R}^m$, $f^{-1}(U)$ is an open subset of $\mathbb{R}^n$.*

Definition 9.12. If $X \subset \mathbb{R}^n$, then a subset $W \subset X$ is *relatively open subset of X (viewing X as a subspace of $\mathbb{R}^n$)* if $W = X \cap U$ for some open subset $U \subset \mathbb{R}^n$.

Theorem 9.13. *If $X \subset \mathbb{R}^n$ and $f\colon X \to \mathbb{R}^m$ is a function, then f is continuous iff for every open subset, $U \subset \mathbb{R}^m$, $f^{-1}(U)$ is a relatively open subset of X.*

We can now formally define the term *homeomorphic*, used in Examples 9.1 and 9.2. First we make some pedantic remarks.

First, if $X \subset \mathbb{R}^n$ is nonempty, then by definition the elements of X consists of n-tuples of real numbers. Hence, by definition, then there is a unique n for which $X \subset \mathbb{R}^n$. Second, if in addition $Y \subset \mathbb{R}^m$, then any map $f\colon X \to Y$ can be viewed as a map $X \to \mathbb{R}^m$;[12] we define f to be continuous if it continuous as a map $X \to \mathbb{R}^m$. The following result is almost immediate.

Proposition 9.14. *Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, and $f\colon X \to Y$. Then f is continuous iff for each relatively open $U \subset Y$, $f^{-1}(U)$ is relatively open in X.*

Definition 9.15. Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$. We say that X and Y are *homeomorphic* if there is a bijection $f\colon X \to Y$ such that f and f^{-1} (hence a bijection $Y \to X$) are both continuous. [Equivalently, for each $U \subset Y$, U is relatively open in f iff $f^{-1}(U)$ is relatively open in X.]

As an EXERCISE, one can verify the claims about homeomorphic sets made in Examples 9.1 and 9.2.

[12] More pedantically, the map $X \to \mathbb{R}^m$ involves a change of the codomain or range of f.

9.4. **A Fork in the Road.** At this point one can either (1) introduce topological spaces, or (2) introduce singular homology groups $H_i^{\mathrm{sing}}(X)$ for $X \subset \mathbb{R}^n$. Ultimately we will define $H_i^{\mathrm{sing}}(X)$ for all topological spaces. My preference in 2025 is to start with (1), since it is important to see a lot of examples of topological spaces before one can really develop intuition about singular i-chains.

But both are possible...

9.5. **Topological Spaces.**

9.5.1. *The Definition of a Topological Space.* According to Proposition 9.14, to decide whether or not $f : X \to Y$ is continuous, we only need to know which are the relatively open sets of X and of Y, and we can forget that $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. This motivates the following definition.

Definition 9.16. A *topological space* is a pair $(X, \mathcal{O})$ consisting of a set, X — the *underlying set* — and a set of subsets of X, $\mathcal{O}$, the *open sets of* X, such that

(1) $\emptyset, X$ are open, i.e., $\emptyset, X \in \mathcal{O}$;
(2) then intersection of any finite number of open sets is again open; and
(3) the union of an arbitrary number of open sets is again open.

When confusion is unlikely, we simply refer to X as the topological space with $\mathcal{O}$ understood (and its elements referred to as *open sets* of X). We also call $\mathcal{O}$ *a topology on* X. We also refer to an element of X as a *point in/of* X.

Example 9.17. The open sets in $\mathbb{R}^n$ are a topology on $\mathbb{R}^n$. If $X \subset \mathbb{R}^n$, then the relatively open subsets of X form a topology on X. (This could be an EXERCISE if you don't know this.)

Definition 9.18. Let X, Y be topological spaces. A map $f : X \to Y$ is *continuous* if for every open set, U, in Y, $f^{-1}(U)$ is open in X.

9.5.2. *New Topological Spaces from Old Ones: Subsets and Products.* There are many ways to get new topological spaces from old ones. Many of these will be crucial to us.

Definition 9.19. Let $(X, \mathcal{O})$ be a topological space, and $X' \subset X$ a subset. A subset $U' \subset X'$ is *relatively open in X' (relative to X' as a subset of $(X, \mathcal{O})$)* if $U' = X' \cap U$ for some open set U of X (i.e., $U \in \mathcal{O}$). We let $\mathcal{O}'$ denote the set of all relatively open sets in X', and refer to $(X', \mathcal{O}')$ or $\mathcal{O}'$ as the *subset topology induced (by $(X, \mathcal{O})$)* or simply *(by X) on X'*.

Example 9.20. Let $X \subset \mathbb{R}^n$. Then the relatively open subsets of X (Definition 9.12) is precisely the topology induced by $\mathbb{R}^n$ on X.

A standard example of the above is the n-sphere

$$\mathbb{S}^n = \{ \mathbf{x} \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1 \}.$$

Example 9.21. If $X = \{p\}$ consists of a single point, p. Then there is only one possible topology on X, namely $\mathcal{O} = \{\emptyset, p\}$.

Example 9.22. Let $X \subset \mathbb{R}^n$ be a finite set of points. Then the relative topology of X is the topology $(X, \mathcal{O})$ where $\mathcal{O}$ consists of all subsets of X. For any set X, the *discrete topology* on X is $\mathcal{O}$ consisting of all subsets of X. If $X = \{1, 2, 3, 4, \ldots\}$, then the relative topology of X (in $\mathbb{R}$) is the discrete topology. By contrast, if

$X = \{0, 1, 1/2, 1/3, 1/4, \ldots\} \subset \mathbb{R}$, then the relative topology of X is *not* the discrete topology: for example, is $\{1/2, 1/4, 1/6, \ldots\}$ an open subset of X?

Definition 9.23. Let $(X_1, \mathcal{O}_1), (X_2, \mathcal{O}_2)$ be two topological spaces. Their *(Cartesian) product* is the topological space $(X_1 \times X_2, \mathcal{O})$, where $\mathcal{O}$ consists of those $U \subset X_1 \times X_2$ that are an union (i.e., an arbitrary union) of sets of the form $U_1 \times U_2$ with $U_1 \in \mathcal{O}_1$ and $U_2 \in \mathcal{O}_2$. Equivalently, U is open if for every $(u_1, u_2) \in U$ there are U_1, U_2, which are respectively open in X_1, X_2, such that $U_1 \times U_2 \subset U$.

Example 9.24. The *d-dimensional torus*, $\mathbb{T}^d$, is the product of d-copies of $\mathbb{S}^1$. The term *the torus* usually refers to $\mathbb{T}^2$.

9.5.3. *New Subspaces from Old Ones: Equivalence Relations: Projective Spaces and the Torus.*

Definition 9.25. Let $(X, \mathcal{O})$, and $\sim$ an equivalence relation on X. We define the *quotient topology (of $(X, \mathcal{O})$ modulo $\sim$)* to be the space $(X/\sim, \mathcal{O}_\sim)$, where $U \in \mathcal{O}_\sim$ iff $U \in \mathcal{O}$, and U is a union of $\sim$ equivalence classes of X.

In general, $\mathcal{O}_\sim$ could be uninteresting, in that it may only contain $\emptyset$ and $X/\sim$. (EXERICSE: Show that this is the case if $X = \mathbb{R}$, $\mathcal{O}$ the open sets of $\mathbb{R}$, and if $\sim$ is the equivalence relation on $\mathbb{R}$ where $x \sim y$ iff $x = y$ or x, y are both rational numbers.) However, equivalence relations are crucial in algebraic topology.

Example 9.26. For $d \geq 0$, we define *real, projective, d-dimensional space*, denoted $\mathbb{RP}^d$, to be the space

$$\mathbb{RP}^d = \bigl(\mathbb{R}^{d+1} \setminus \{\mathbf{0}\}\bigr) \Big/ \bigl(\mathbb{R} \setminus \{0\}\bigr),$$

i.e., $(\mathbb{R}^{d+1} \setminus \{0\})/\sim$, where $x \sim y$ iff they are scalar multiples of each other. Equivalently, $\mathbb{RP}^d$ can be identified with the set of lines in $\mathbb{R}^{d+1}$, or with $\mathbb{S}^d/\sim$ under the equivalence relation $x \sim y$ iff $x = \pm y$. Similarly for complex, projective space,

$$\mathbb{CP}^d = \bigl(\mathbb{C}^{d+1} \setminus \{\mathbf{0}\}\bigr) \Big/ \bigl(\mathbb{C} \setminus \{0\}\bigr)$$

and lines in $\mathbb{C}^{d+1}$.

Example 9.27. Generalizing projective space, if we fix $1 \leq m \leq n$, the m-dimensional subspaces of $\mathbb{R}^n$ is a topological space known as a *Grassmannian*, denoted $\mathrm{Grass}(m, n)$. One way to describe these as topological spaces is as follows: let $\mathrm{Bases}(m, n)$ denote the sequences of m linearly independent vectors $(\mathbf{u}_1, \ldots, \mathbf{u}_m) \in \mathbb{R}^n$; hence

$$\mathrm{Bases}(m, n) \subset \bigl(\mathbb{R}^n\bigr)^m.$$

One then notes that the set of invertible $m \times m$ matrices, denoted $\mathrm{GL}(m)$ ("general linear group"), acts on $\mathrm{Bases}(m, n)$, and one sets

$$\mathrm{Grass}(m, n) = \mathrm{Bases}(m, n)/\mathrm{GL}(m).$$

A more explicit way to construct $\mathrm{Grass}(m, n)$ is to give "coordinates" on this space: namely, many m-dimensional subspaces can be written as the rowspace of an $m \times n$ block matrix of the form

$$A = [I|M],$$

where I is the $m \times m$ identity matrix, and M is an arbitrary $m \times (n - m)$ matrix; moreover the set of m-dimensional subspaces of this form is an open subset of $\mathrm{Grass}(m,n)$. Similarly if we exchange the columns of A, and any element of $\mathrm{Grass}(m,n)$ is the rowspace of some column permutation of a matrix $[I|M]$. This describes $\mathrm{Grass}(m,n)$ as the union of open sets, each of which is $m(n-m)$-dimensional *manifold*. We have $\mathrm{Grass}(1,n+1)=\mathbb{RP}^n$, so Grassmannians are generalizations of projective spaces.

Example 9.28. Consider the $X=[0,1]\times[0,1]\in\mathbb{R}^2$. By identifying the boundary of X is various ways, the resulting spaces $X/\sim$ yield (a simple to describe space that is isomorphic to) the torus $\mathbb{T}=\mathbb{T}^2$, the real projective plane $\mathbb{RP}^2$, and the *Klein bottle*. See Hatcher's textbook [Hat02], Section 2.1 (page 102).

9.5.4. *New Subspaces from Old Ones: Equivalence Relations: Collapsing Along a Subset, Cones, Wedge Sums, Etc.*

Definition 9.29. Let X be a topological space, and $A\subset X$. We use X/A to denote the space $X/\sim$ where $x\sim x'$ iff $x=x'$ or $x,x'\in A$. We call this the *collapse of* X *along* A.

Example 9.30. Let X be a topological space. We define the *topological cone of* X to be

$$\mathrm{ConeTop}(X)\stackrel{\mathrm{def}}{=}\bigl(X\times[0,1]\bigr)/X\times\{1\}. \tag{47}$$

If $X\subset\mathbb{R}^n$, we define the *cone of* X *(as a subspace of* $\mathbb{R}^n$*)* to be the subspace of $\mathbb{R}^{n+1}$ given by

$$\begin{aligned}\mathrm{Cone}_{\mathbb{R}^n}(X)&\stackrel{\mathrm{def}}{=}\bigcup_{\mathbf{x}\in X}\mathrm{Conv}\bigl((\mathbf{x},0),(\mathbf{0},1)\bigr)\\&=\Bigl\{(t\mathbf{x},1-t)\ \Bigm|\ \mathbf{x}\in X,\ 0\le t\le 1\Bigr\}.\end{aligned}$$

For each $X\subset\mathbb{R}^n$, there is a natural bijection of sets

$$f\colon\mathrm{ConeTop}(X)\to\mathrm{Cone}_{\mathbb{R}^n}(X), \tag{48}$$

given by $(x,t)\mapsto((1-t)\mathbf{x},t)$ for $0\le t\le 1$ (thereby taking $X\times\{1\}$ to $(\mathbf{0},1)$). It turns out that f is continuous, and if $X=|K|$ where K is a (finite) simplicial complex in $\mathbb{R}^n$, then f^{-1} is continuous; hence, when $X=|K|$, the two notions of a cone are homeomorphic. However, the two notions are not homeomorphic for general $X\subset\mathbb{R}^n$.

Example 9.31. Say that $(X,\mathcal{O})$ and $(X',\mathcal{O}')$ are topological spaces with X,X' disjoint. Then we define their *union* to be the space $(X\cup X',\mathcal{O}'')$ where $\mathcal{O}''$ consists of all unions of one set of $\mathcal{O}$ with another of $\mathcal{O}'$.

The "disjoint union" will be fundamental to much of what we do.

Example 9.32. The *disjoint union* of two sets X and Y, denoted $X\amalg Y$, intuitively refers to a union of two disjoint sets, one in bijection with X, the other in bijection with Y; it is *not* a unique set (but it is "unique up to unique isomorphism," as we shall explain). Hence if X and Y are disjoint sets, $X\amalg Y$ can be viewed as $X\cup Y$. However, one often has that X,Y intersect (think of vertices and edges of a graph that have been enumerated $\{1,\dots,n\}$ and $\{1,\dots,m\}$) or even $X=Y$; one typical convention is to take $X\amalg Y$ to mean $X\times\{1\}\cup Y\times\{2\}$. Formally we define

$X \amalg Y$ to be a type of "limit"[13], which amounts to a triple (S, f, g) of a set S and injections $f: X \to S$ and $g: Y \to S$ such that $f(X), g(Y)$ are disjoint and their union is all of S; we usually just use $X \amalg Y$ to denote S, with f, g understood. We similarly define an arbitrary disjoint union, e.g., $\amalg_{\alpha \in A} X_\alpha$, which we can take to mean $\bigcup_{\alpha \in A} X_\alpha \times \{\alpha\}$.[14] If X, Y are topological spaces, then their *disjoint union* is the set $X \amalg Y$ where U is open iff $U = U_1 \cup U_2$ where U_1 is open in X (viewing $U_1 \subset X$ as lying in $X \amalg Y$) and similarly U_2 is open in Y.

Example 9.33. The *wedge sum of X and Y* is the space

$$X \vee Y = (X \amalg Y)/\{x, y\},$$

i.e., the disjoint union of X and Y, where x and y are identified. (This generally depends on the choice of x, y.)

9.5.5. *Metric Spaces (Optional, Maybe an EXERCISE).* For those who know what is meant by a *metric space*, a metric space gives rise to a topological space, and most spaces of interest to us will arise in this way.

[However, in these notes will won't have a particular interest in metric spaces, at least not per se.]

Example 9.34. Let (X, ρ) be a metric space[15]. We define $U \subset X$ to be open if for each $u \in U$, U contains a ball of some positive radius about u (open or closed ball, it doesn't matter). This gives a topology on X.

A topological space arising as such from a metric space is called *metrizable*; any such topological space is separated (also known as Hausdorff or "T_2"): if $x \neq y$, then there are two disjoint open subsets, one containing x and the other containing y.

EXERCISE: Let S be an infinite set, and say that $U \subset S$ is *cofinite* if $S \setminus U$ is finite. Show that the cofinite sets of S form a topology. Is this topological space metrizable? [If $S = \mathbb{F}$ is an algebraically closed field, then this is the Zariski topology on $\mathbb{A}^1(\mathbb{F})$, the 1-dimensional affine line over $\mathbb{F}$, the fundamental topology of algebraic geometry.]

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[13]Namely a disjoint union is any initial element in the category whose objects are (S, f, g), consisting of a set S and maps $f: S \to X$ and $g: S \to Y$. Hence if (S, f, g) is an initial object, and (S', f', g') is another other object, there must be a unique morphism $(S, f, g) \to (S', f', g')$, i.e., a unique map of sets $\phi: S \to S'$ respecting the maps in the evident sense, i.e., $f'\phi = f$ and $g'\phi = g$. We easily see that an initial object does exist, such as $(X \times \{1\} \cup Y \times \{2\}, f, g)$ where f takes $X \times \{1\}$ to X in the evident bijection, and similarly for $g: Y \times \{2\} \to Y$. By definition, if an initial object exists, then it is unique up to unique isomorphism: i.e., here, if (S, f, g) and (S', f', g') are two initial objects of this category, then there is a unique morphism $(S, f, g) \to (S', f', g')$, meaning a map $\phi: S \to S'$ (which turns out to necessarily be a bijection), which respects the maps in an evident sense. If all this seems like a bunch of abstract nonsense, then you need to understand why some reasonable notions of "moduli spaces" turn out "wrong," unless you consider them as functors via category theory, and these functors — fine moduli spaces — turn out to be exactly the "right" definition.

[14]Some authors use "the disjoint union of X and Y" in a narrow sense, namely to imply that X and Y are disjoint sets, and then take $X \amalg Y$ to mean $X \cup Y$; however, this will not work for us, because we will build Δ-complexes, such as graphs with multiple edges and self-loops, using disjoint unions where the same set appears many times.

[15]If you don't know this definition, you can look this up and see some examples. In brief, X is a set, and $\rho: X \times X \to \mathbb{R}$ is a function such that: (1) $\rho(x, y) \geq 0$, with equality iff $x = y$; (2) $\rho(x, y) = \rho(y, x)$; and (3) ρ satisfies the triangle inequality.

9.5.6. *More on Cones and Suspensions of Topological Spaces.*

Example 9.35. Let X be a topological space. Just as we've defined the *topological cone of X* in (47), we can similarly define the *topological suspension of X* to be:

$$\mathrm{Suspension}(X) \stackrel{\mathrm{def}}{=} \bigl(X \times [-1,1]\bigr)/\sim$$

where $\sim$ "collapses $X \times \{-1\}$ to a point and collapses $X \times \{1\}$ to a point," i.e., $(x,t) \sim (x',t')$ iff either $(x,t) = (x',t')$ or $t = t' = \pm 1$. Compare [Hat02], pages 8 and 9.

9.6. **Maps from Simplices, Ordered Simplices.**

Remark 9.36. At this point (or earlier?), if X, Y are topological spaces, a *map* $f\colon X \to Y$ means "continuous map," unless otherwise mentioned.

If X is a topological space, we will soon speak of the *singular homology groups* of X, and a Δ-*complex* structure on X. Both notions are built using maps $S \to X$ where S is a simplex. However, for numerous reasons we will need to remember an ordering of the vertices of S.

The following definitions are a bit pedantic, but omitting them might risk confusion.

Definition 9.37. Let $n, N, N' \geq 0$ be integers. An *ordered n-simplex (in $\mathbb{R}^N$)* refers to a sequence $\mathfrak{S} = (\mathbf{a}_0, \ldots, \mathbf{a}_n)$ of vectors of $\mathbb{R}^N$ that are in general position (therefore $N \geq n$). [$\mathfrak{S}$ is the Fraktur letter S.] We refer to

$$S = \mathrm{Conv}(\mathbf{a}_0, \ldots, \mathbf{a}_n)$$

as the *underlying simplex* of $\mathfrak{S}$, and $\mathbf{a}_0, \ldots, \mathbf{a}_n$ as the *sequence of ordered vertices of* $\mathfrak{S}$. If $\mathbf{t} = (t_0, \ldots, t_n)$ is a stochastic vector (i.e., non-negative reals that sum to 1), we use the notation

$$\mathbf{t}_{\mathfrak{S}} = (t_0, \ldots, t_n)_{\mathfrak{S}} \stackrel{\mathrm{def}}{=} t_0\mathbf{a}_0 + \ldots + t_n\mathbf{a}_n \in \mathbb{R}^N,$$

and recall that each $s \in S$ can be written uniquely as such (with $\mathbf{t}$ stochastic), and that $\mathbf{t} = (t_0, \ldots, t_n)$ is called the *barycentric coordinate of s*. If $\mathfrak{S}' = (\mathbf{a}'_0, \ldots, \mathbf{a}'_n)$ is another ordered n-simplex in $\mathbb{R}^{N'}$, then there is a unique isomorphism $S \to S'$ given by $\mathbf{t}_{\mathfrak{S}} \mapsto \mathbf{t}_{\mathfrak{S}'}$ which we call the *canonical isomorphism (from S to S')*. If confusion is unlikely, we write S instead of $\mathfrak{S}$, understanding that S comes with an ordering of its vertices.

Both Δ-*complexes* and *singular homology* are usually defined by fixing for each $n \geq 0$ a single "standard" ordered n-simplex.

Definition 9.38. Let $n \geq 0$ be an integer. The *standard (ordered) n-simplex*, denoted $\mathfrak{D}^n$, refers to the ordered n-simplex $(\mathbf{e}_1, \ldots, \mathbf{e}_{n+1})$ in $\mathbb{R}^{n+1}$, where $\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}$ are the standard basis vectors of $\mathbb{R}^{n+1}$. We write Δ^n for the associated simplex

$$\Delta^n = \mathrm{Conv}(\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}) \subset \mathbb{R}^{n+1}.$$

If $\mathfrak{S}$ is any ordered n-simplex, and $f\colon S \to X$ is any map from the underlying simplex of $\mathfrak{S}$ to any topological space, X, the *standardized version of f*, denoted $\mathrm{Stand}(f)$ refers to the map $\Delta^n \to X$ obtained by composing the canonical isomorphism $\Delta^n \to S$ with f.

One novelty of the standard ordered n-simplex, $\mathfrak{D}^n$ is that its barycentric coordinate, $\mathbf{t}_{\mathfrak{D}^n}$ for elements of this simplex actually *equals* $\mathbf{t} \in \mathbb{R}^{n+1}$.

Definition 9.39. Let X be a topological space, and let $\sigma\colon \Delta^n \to X$ be a map. We define the *boundary of* σ, denoted $\partial_n \sigma$, to be the formal linear combination of maps

$$\partial_n \sigma \stackrel{\text{def}}{=} \sum_{j=0}^{n} (-1)^j \mathrm{Stand}\bigl(\sigma|_{(\mathbf{e}_1,\ldots,\widehat{\mathbf{e}_j},\ldots,\mathbf{e}_{n+1})}\bigr), \tag{49}$$

i.e., the alternating sum of the standardized maps obtained by restricting σ to its $(n-1)$-dimensional faces,

$$(\mathbf{e}_1,\ldots,\widehat{\mathbf{e}_j},\ldots,\mathbf{e}_{n+1}),$$

each such face being an ordered simplex in this way.

Therefore the right-hand-side of (49) is an alternating sum of maps $\Delta^{n-1} \to X$.

Warning: At this point we will usually write Δ^n instead of $\mathfrak{D}^n$, since this is the common notation (e.g., in Hatcher's textbook [Hat02]).

9.7. **Singular Homology 1: Simplicial Singular Homology.** At this point we may as well describe one way to build singular homology, since we have all the necessary definitions. We caution the reader that there are two common ways to do this, each with one basic advantage over the other (see later); the first is called *(simplicial) singular homology*, the second called *(cubical) singular homology*. To understand singular homology, it is easiest to (eventually) understand both constructions. [There are many variants of these two constructions.]

Definition 9.40. Let X be a topological space. For each integer $n \geq 0$, we define the *(simplicial) singular n-chains on* X, denoted $\mathcal{C}_n(X)$ (or sometimes $\mathcal{C}_n^{\mathrm{sing}}(X)$ or $\mathcal{C}_n^{\mathrm{sing(simp)}}(X)$ for clarity) to be the set of formal linear combinations of maps $\Delta^n \to X$. We define the maps

$$\partial_n \colon \mathcal{C}_n^{\mathrm{sing}}(X) \to \mathcal{C}_{n-1}^{\mathrm{sing}}(X)$$

as the linear extension of ∂_n of (49) (of Definition 9.39).

Remark 9.41. (WARNING...) It is crucial to understand that the map ∂_n **immediately** extends from (49) to a map defined on all of $\mathcal{C}_n^{\mathrm{sing}}(X)$: this is because in defining $\mathcal{C}_n^{\mathrm{sing}}(X)$, we **do not** identity a map $\Delta^n \to X$ with any other map $\Delta^n \to X$ (say obtained by exchanging the order of vertices). In other words, two maps $\sigma\colon \Delta^n \to X$ and $\sigma'\colon \Delta^n \to X$ are considered the same in $\mathcal{C}_i^{\mathrm{sing}}(X)$ iff σ and σ' are identical maps, agreeing on all of Δ^n. This contrasts with the i-chains, $\mathcal{C}_i(\mathsf{K})$, of an abstract simplicial complex, where we identify the 1-chain $[A,B]$ with $-[B,A]$ (Subsection 4.1), and similarly for permuting the vertices of higher chains (Subsections 4.4 and 4.5).

Remark 9.42. The term "singular" in *singular homology* refers to the fact that the image of a map $\Delta^n \to X$ can be rather pathological, and this image is not generally homeomorphic to something nice (like a simplex). Note that the set of maps $\Delta^n \to X$ is a basis of $\mathcal{C}_n^{\mathrm{sing}}(X)$, and the later is therefore a "vastly" infinite dimensional space.

Remark 9.43. One type of very degenerate map $\Delta^n \to X$ is the map taking each element of Δ^n to a single point $x \in X$. We will soon see that these maps are *crucial* to getting singular homology to work out.

EXERCISE: A *path* in a topological space, X, is a (continuous) map $p\colon [0,1] \to X$, and we say that p *runs* (or *is*) from $p(0)$ to $p(1)$. We say that $x, x' \in X$ are *(path) connected* if there is a path in X from x to x'; show that being (path) connected is an equivalence relation. The equivalence classes are called the *(path) connected components of* X.

Remark 9.44. Had we first discussed the *fundamental group*, $\pi_1(X)$, of a topological space X, we would see how paths in X can be used to get a remarkable amount of information about X. Of course, $[0,1]$ can be viewed as an ordered 1-simplex with the order $0, 1$, and hence a path is equivalent to a map $\Delta^1 \to X$. This may make it more plausible that singular homology is working in a reasonable setup.

We easily verify that

$$\cdots \xrightarrow{\partial_3} \mathcal{C}_2^{\rm sing}(X) \xrightarrow{\partial_2} \mathcal{C}_1^{\rm sing}(X) \xrightarrow{\partial_1} \mathcal{C}_0^{\rm sing}(X) \to 0 \tag{50}$$

is a chain.

Definition 9.45. Let X be a topological space. We define the *singular homology groups* and *singular Betti numbers* of X to be those of (50), i.e.,

$$H_i^{\rm sing}(X) \stackrel{\rm def}{=} \ker(\partial_i) \,/\, \mathrm{Image}(\partial_{i+1}), \quad \beta_i^{\rm sing}(X) \stackrel{\rm def}{=} \dim\bigl(H_i^{\rm sing}(X)\bigr).$$

Although the $\mathcal{C}_i^{\rm sing}(X)$ are "vastly" infinite dimensional spaces, the homology groups above turn agree with simplicial homology on simplicial complexes.

9.8. **Singular Homology 2: Why Do We Care About Singular Homology?** Here are some of the main points of singular homology of interest to us.

(1) The groups $H_i^{\rm sing}(X)$ are defined for any topological space, X.
(2) If $f\colon X \to Y$ is a map of topological spaces, then clearly f gives a map from each $\sigma\colon \Delta^n \to X$ to a map $f \circ \sigma\colon \Delta^n \to Y$, which induces maps $f_{\#,i}$ from $\mathcal{C}_i^{\rm sing}(X) \to \mathcal{C}_i^{\rm sing}(Y)$ and therefore maps $f_{*,i}$ from $H_i^{\rm sing}(X) \to H_i^{\rm sing}(Y)$. It is generally much easier to work with maps from one topological space to another than it is to work with maps of simplicial complexes. Here's why:
   (a) If K, K' are simplicial complexes, the easiest way to map $f\colon |K| \to |K'|$ in a way that gives maps $H_i(\mathsf{K}) \to H_i(\mathsf{K}')$ is to insist that f be a *simplicial map*, i.e., that f map the vertex sets $V(K) \to V(K')$ and that for each $X \in K$, f maps X to an $X' \in K$ via the map

   $$t_0\mathbf{a}_0 + \cdots + t_n\mathbf{a}_n \mapsto t_0 f(\mathbf{a}_0) + \cdots + t_n f(\mathbf{a}_n).$$

   Simplicial maps are a very limited type of map. For example:
      (i) there is a counter-clockwise rotation by θ map taking $\mathbb{S}^1 \to \mathbb{S}^1$ for any $\theta \in [0, 2\pi)$. However, if K is a regular k-gon in $\mathbb{R}^2$, then only rotations by a multiple of $2\pi/k$ are simplicial maps.
      (ii) There is a simple embedding of $\mathbb{S}^n$ in $\mathbb{S}^{n+1}$, and it is easy to see that $\mathbb{S}^{n+1}$ is isomorphic to the suspension of $\mathbb{S}^n$ (in either way of defining a suspension). These facts before much more tedious when we replace spheres by simplicial complexes that are homeomorphic to spheres.

(iii) Two maps f, g from $X \to Y$ are *homotopic* if there is a map $F\colon X \times [0,1] \to Y$ where $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$; we will often use homotopies from one map to another. It is more difficult to build homotopies for simplicial complexes, for numerous reasons, if we restrict to simplicial maps.

(iv) A good example of homotopic maps to keep in mind was mentioned in class: for any K, $\mathsf{L} = \mathrm{Cone}_P(\mathsf{K})$ has the same homology groups as a point; the proof is actually based on a homotopy between the identity map, f, on $|L|$ to the map $g\colon |L| \to P$.

(v) In fairness, if f, g are maps of abstract simplicial complexes, $\mathsf{K} \to \mathsf{L}$, one can define a notion of f and g being *contiguous* (e.g., [Mun84], page 67),[16] which implies that the induced maps f_*, g_* from $H_i(\mathsf{K}) \to H_i(\mathsf{L})$ are equal. This is a sort of "substitute" for homotopy equivalence of topological spaces. For example, we see that the (simplicial) identity map, f, on $\mathrm{Cone}_P(\mathsf{K})$ is contiguous with the (simplical) map $g\colon \mathrm{Cone}_P(\mathsf{K}) \to \{P\}$, which proves that $\mathrm{Cone}_P(\mathsf{K})$ and $\{P\}$ have the same simplicial homology groups.[17]

(b) If K, L are simplicial complexes, one can take a general continuous map $h\colon |K| \to |L|$ and obtain maps $h_\#\colon H_i(\mathsf{K}) \to H_i(\mathsf{L})$ in simplicial homology, but the proof is a bit delicate, namely:

(i) We say that a simplicial map $f\colon \mathsf{K} \to \mathsf{L}$ is a *simplicial approximation* of h if for each vertex $v \in V(\mathsf{K}) = V(K)$ we have that $h(\mathrm{Star}(v)) \subset \mathrm{Star}(f(v))$ (see [Mun84], Section 14, Chapter 2, bottom of page 80); one then proves that any two simplicial approximations of h are contiguous, so one gets a uniquely determined map of simplicial homology, h_*, $H_i^{\mathrm{simp}}(\mathsf{K}) \to H_i^{\mathrm{simp}}(\mathsf{L})$.

(ii) The complex K has a *barycentric subdivision*, $\mathbf{sd}(K)$, where we define each n-simplex in K to 2^n subsimplices according to its barycentric coordinates $(t_0, \ldots, t_n)$, and according to whether or not each t_i is $> 1/2$, $= 1/2$, or $< 1/2$. Of course, $|\mathbf{sd}(K)| = |K|$. One can prove that for any $h\colon |K| \to |L|$, there is a *subdivision*, K' of K (which we can take to be the M-th iterated barycentric subdivision of K for some M) such that h, viewed as a map $|K'| \to |L|$ has a simplicial approximation (see [Mun84] Theorem 16.1, Section 16, Chapter 2, page 89).

So a map $h\colon |K| \to |L|$ of topological spaces does give a map $h_{*,i}\colon H_i(\mathsf{K}) \to H_i(\mathsf{L})$, but this is rather technical. We feel that working with topological spaces tends to be a simpler framework to develop intuition.

(3) A particular case of (2) is that if $f\colon X \to Y$ is a homeomorphism, the maps $f_{*,i}$ and $(f^{-1})_{*,i}$ provide isomorphisms $H_i^{\mathrm{sing}}(X) \to H_i^{\mathrm{sing}}(Y)$.

(4) If $X = |K|_{\mathrm{geom}}$ is the geometric realization of a simplicial complex K, whose underling abstract complex is K, any i-simplex $(u_0, \ldots, u_i)$ of K

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[16] Namely, two simplicial maps f, g from $\mathsf{K} \to \mathsf{L}$ are *contiguous* if for every $S \in \mathsf{K}$, $f(S) \cup g(S) \in \mathsf{L}$. [Of course, $f(S)$ is of smaller size than S when $f|_S$ is not injective, and similarly for $g(S)$, and contiguity allows for $f(S)$ and $g(S)$ to intersect.]

[17] Indeed, if $S \in \mathsf{K}$, then $f(S) = S$ when S is the identity on $\mathrm{Cone}_P(\mathsf{K})$, and $g(S) = \{P\}$; by definition of the cone, $S \cup P$ always lies in $\mathrm{Cone}_P(\mathsf{K})$.

corresponds to an ordered i-simplex $S \in K$, which is therefore entirely contained in $|K| = X$. Hence the i-simplex $(u_0, \dots, u_i)$ of K gives rise to a map $\Delta^i \to |K| = X$ via barycentric coordinates, which we denote $\xi(u_0, \dots, u_i)$. The main theorem we need to know is that the map

$$(u_0, \dots, u_i) \mapsto \xi(u_0, \dots, u_i)$$

extends to an isomorphism

$$\Xi\colon H_i^{\mathrm{simp}}(\mathsf{K}) \to H_i^{\mathrm{sing}}(|K|) \tag{51}$$

(which takes $[u_0, \dots, u_i]$ to $\xi(u_0, \dots, u_i)$, as it lies in $H_i^{\mathrm{sing}}(|K|)$). This takes some work to prove, but this work is very rewarding and illuminating (this proof starts in Hatcher [Hat02] on page 127 there, Section 2.1). We'll describe some of the steps in the next subsection.

(5) We say that two maps f, g from $X \to Y$ are homotopic, then the proof that f_* and g_* give the same maps $H_i^{\mathrm{sing}}(X) \to H_i^{\mathrm{sing}}(Y)$ is not difficult (see Hatcher [Hat02], page 111, Theorem 2.10). The argument is even simpler when we introduce *cubical singular homology*, due to the fact that the product of a "cube" with $[0,1]$ is again a "cube;" see below.

(6) We say that two spaces, X, Y are *homotopy equivalent* if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that the map $X \to X$ given by gf is homotopic to id_X (the identity map on X) and fg is homotopic to id_Y. We easily see that (1) homotopy equivalence is an equivalence relation on spaces, and (2) if X, Y are homotopy equivalent, then (in view of the facts above)

$$f_{*,i}\colon H_i^{\mathrm{sing}}(X) \to H_i^{\mathrm{sing}}(Y)$$

is an isomorphism.

9.9. **Retractions and Brouwer's Fixed Point Theorem.** At this point we know that $\mathbb{S}^n$ is isomorphic to the suspension of $\mathbb{S}^{n-1}$, and hence, by induction on $n \geq 1$, we have the Betti numbers of $\mathbb{S}^n$, $\beta_i(\mathbb{S}^n)$, equal 1 for $i = 0, n$, and 0 if $i \neq 0, n$. Since $\mathbb{S}^0$ consists of two points, we have $\beta_i(\mathbb{S}^0)$ equals 2 for $i = 0$, and 0 otherwise.

In this subsection we will prove Brouwer's fixed-point theorem.

Theorem 9.46 (Brouwer's fixed point theorem)**.** *Let $n \geq 1$, and let $\mathbb{D}^n$ be the unit disc in $\mathbb{R}^n$, i.e.,*

$$\mathbb{D}^n = \{\mathbf{x} \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}.$$

Then any continuous map $f\colon \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point, i.e., for some $x \in \mathbb{D}^n$ we have $f(x) = x$.

Definition 9.47. If X is a topological space, then we say that a subset $A \subset X$ is a *retraction of* X if there exists a map $f\colon X \to A$ (the *retraction map*) such that $f|_A$ (i.e., f restricted to A) is the identity map, and $f(X) \subset A$ (therefore $f(X) = A$).

Lemma 9.48. *If $A \subset X$ is a retraction of X, then for all i we have $\beta_i(A) \leq \beta_i(X)$.*

Proof. Let $f\colon X \to A$ be a retraction map, and let $\iota\colon A \to X$ be the inclusion map. Then $f\iota = \mathrm{id}_A$. Hence $(f\iota)_*$ is the identity map, and this is a composition of maps

$$H_i(A) \xrightarrow{\iota_*} H_i(X) \xrightarrow{f_*} H_i(A).$$

Hence the map $H_i(X) \to H_i(A)$ is surjective, and hence the dimension of $H_i(X)$ must be at least that of $H_i(A)$. $\square$

Proof of Brouwer's fixed point theorem. Say that $f\colon \mathbb{D}^n \to \mathbb{D}^n$ has no fixed point. For each $\mathbf{x}$ in the interior of $\mathbb{D}^n$, the triangle inequality easily implies that $\mathbb{D}^n$ is strictly convex: i.e., for any $\mathbf{x}_1 \neq \mathbf{x}_2 \in \mathbb{D}^n$, for any $0 < t < 1$, $t\mathbf{x}_1 + (1-t)\mathbf{x}_2$ is in the interior of $\mathbb{D}^n$. This convexity of $\mathbb{D}^n$ implies that there is a unique point on $\mathbb{S}^{n-1}$ that meets the ray beginning at $\mathbf{x}$ and pointing in the direction of (the nonzero vector) $\mathbf{x} - \mathbf{f}(\mathbf{x})$ (draw a picture); therefore

$$\mathbf{g}(\mathbf{x}) = \{\mathbf{x} + \bigl(\mathbf{x} - \mathbf{f}(\mathbf{x})\bigr)t \mid t \geq 0\} \cap \mathbb{S}^{n-1}$$

defines a map from $\mathbb{D}^n$ to $\mathbb{S}^{n-1}$.

We claim g is continuous: to prove this, it is easiest to use sequences: let $\mathbf{x} \in \mathbb{D}^n$, and $\mathbf{x}_1, \mathbf{x}_2, \ldots$ a sequence in $\mathbb{D}^n$ whose limit is $\mathbf{x}$. By definition, there are unique non-negative $t_1, t_2, \ldots$ such that

$$\mathbf{g}(\mathbf{x}_i) = \mathbf{x}_i + \bigl(\mathbf{x}_i - \mathbf{f}(\mathbf{x}_i)\bigr)t_i,$$

and therefore

$$\Bigl|\mathbf{x}_i + \bigl(\mathbf{x}_i - \mathbf{f}(\mathbf{x}_i)\bigr)t_i\Bigr| = 1.$$

Similarly for a unique $t \geq 0$ we have

$$g(\mathbf{x}) = \mathbf{x} + \bigl(\mathbf{x} - \mathbf{f}(\mathbf{x})\bigr)t$$

and therefore

$$\Bigl|\mathbf{x} + \bigl(\mathbf{x} - \mathbf{f}(\mathbf{x})\bigr)t\Bigr| = 1.$$

Let us prove that $t_i \to t$ as $i \to \infty$.

For every $\epsilon > 0$ we have that $\mathbf{x} + \bigl(\mathbf{x} - \mathbf{f}(\mathbf{x})\bigr)(t+\epsilon)$ lies outside of $\mathbb{D}^n$, and $\mathbf{x} + \bigl(\mathbf{x} - \mathbf{f}(\mathbf{x})\bigr)(t-\epsilon)$ lies in the interior of $\mathbb{D}^n$. Hence there is a $\delta > 0$ such that,

$$\Bigl|\mathbf{x} + \bigl(\mathbf{x} - \mathbf{f}(\mathbf{x})\bigr)(t+\epsilon)\Bigr| \geq 1 + \delta$$

and

$$\Bigl|\mathbf{x} + \bigl(\mathbf{x} - \mathbf{f}(\mathbf{x})\bigr)(t-\epsilon)\Bigr| \leq 1 - \delta$$

By continuity of $\mathbf{f}$, it follows that for sufficiently large i,

$$\Bigl|\mathbf{x}_i + \bigl(\mathbf{x}_i - \mathbf{f}(\mathbf{x}_i)\bigr)(t+\epsilon)\Bigr| \geq 1 + \delta/2,$$

and

$$\Bigl|\mathbf{x}_i + \bigl(\mathbf{x}_i - \mathbf{f}(\mathbf{x}_i)\bigr)(t-\epsilon)\Bigr| \leq 1 - \delta/2.$$

Hence for i sufficiently large, $t - \epsilon < t_i < t + \epsilon$, and therefore as $i \to \infty$, $t_i \to t$.

Since as $i \to \infty$, $t_i \to t$, it follows that

$$\begin{aligned}\lim_{i\to\infty} \mathbf{g}(\mathbf{x}_i) &= \lim_{i\to\infty} \Bigl(\mathbf{x}_i + \bigl(\mathbf{x}_i - \mathbf{f}(\mathbf{x}_i)\bigr)t_i\Bigr) \\ &= \mathbf{x} + \bigl(\mathbf{x} - \mathbf{f}(\mathbf{x})\bigr)t\Bigr) = \mathbf{g}(\mathbf{x});\end{aligned}$$

hence $\mathbf{g}$ is continuous.

Note also that if $\mathbf{x} \in \mathbb{S}^{n-1}$, then $|\mathbf{x}| = 1$ and therefore $g(\mathbf{x}) = \mathbf{x}$. Hence g is a continuous function $g\colon \mathbb{D}^n \to \mathbb{S}^{n-1}$ whose restriction to $\mathbb{S}^{n-1}$ is the identity map. Hence $\mathbb{S}^{n-1} \subset \mathbb{D}^n$ is a retraction of $\mathbb{D}^n$. But we know that,

$$\text{if } n \geq 2, \text{ then } \beta_{n-1}(\mathbb{S}^{n-1}) = 1 \text{ and } \beta_{n-1}(\mathbb{D}^n) = 0,$$

and
$$\text{if } n=1, \text{ then } \beta_{n-1}(\mathbb{S}^{n-1})=2 \text{ and } \beta_{n-1}(\mathbb{D}^n)=1,$$
and together these imply that $\mathbb{S}^{n-1}$ cannot be a retraction of $\mathbb{D}^n$. □

Corollary 9.49. *Let X be a topological space that is homeomorphic to $\mathbb{D}^n$, Then any function $f\colon X\to X$ has a fixed point.*

The proof is that if $g\colon X\to\mathbb{D}^n$ is a homeomorphism, and f has no fixed point, then gfg^{-1} is a map $\mathbb{D}^n\to\mathbb{D}^n$ without a fixed point.

To apply this corollary it will be useful to know that various topological spaces are isomorphic to $\mathbb{D}^n$. In Exercise B.20 we will prove the following fact.

Definition 9.50. Let $X\subset\mathbb{R}^n$ be a subset, and $x_0\in X$. We say that X is *star-shaped at x_0* if (1) x_0 lies in the interior of X, (2) X is closed and bounded (i.e., X is compact), and (3) for each $\mathbf{x}\neq\mathbf{0}$, there is a unique $t>0$ such that $\mathbf{x}_0+t\mathbf{x}\in\partial X$, where ∂X is the boundary of X (i.e., X minus its interior).[18]

Theorem 9.51. *Let X be star-shaped at $x_0\in X$. Then: (1) X is homeomorphic to $\mathbb{D}^n$, and (2) ∂X is homeomorphic to $\mathbb{S}^{n-1}$.*

For the proof, see Exercise B.20.

Remark 9.52. Our proof of Theorem 9.46 is especially convenient because $\mathbb{S}^{n-1}=\partial\mathbb{D}^n$, the boundary of $\mathbb{D}^n$, consists of precisely those $\mathbf{x}$ with $|\mathbf{x}|=1$. The proof would not work if we replaced $\mathbb{D}^n$ with an X homeomorphic to $\mathbb{D}^n$ such that X is not convex; it would work, but be a bit more awkward, had we worked with an n-simplex instead of $\mathbb{D}^n$. Hence working with topological spaces allows us the freedom to prove various theorems, like Theorem 9.46, with a particularly convenient space, such as $\mathbb{D}^n$, within a homeomorphism class of topological spaces.

In Subsection B.3, we give some applications of the Brouwer fixed point theorem, including a proof of the Perron-Frobenius theorem and the existence of a Nash equilibrium.

9.10. **Singular Homology 3: The Isomorphism between Simplicial Homology and Singular (Simplicial) Homology.** Let us describe some basic steps needed to show that
$$[u_0,\ldots,u_i]\mapsto\xi(u_0,\ldots,u_i)$$
extends to an isomorphism (51). In doing so we will develop some rewarding intuition regarding homology.

(1) First, we claim that if $X=\{p\}$ is a single point, then $\beta_0(X)=1$ and $\beta_i(X)=0$ for $i\geq 1$. This may seem like a trivial step, but this will shed some light on the setup of singular (simplicial) homology. Moreover, this will shed some crucial light on how *cubical singular homology* is defined below, which requires a somewhat awkward convention in order for $\beta_i(\{p\})$ to be 1 if $i=0$ and 0 for $i\geq 1$. So for $X=\{p\}$, there is exactly one

[18]Munkres' textbook [Mun84] defines a weaker notion of *star-convex* for an open, bounded subset $U\in\mathbb{R}^n$, namely that for any $\mathbf{x}\in U$, $\mathrm{Conv}(\mathbf{0},\mathbf{x})\subset U$; see Exercise 5 there, end of Section 1, page 7. For this weaker definition, $\overline{U}$, i.e., the closure of U, need not be star-shaped at $\mathbf{0}$.

singular n-simplex, namely the map $\sigma_n\colon \Delta^n \to \{p\}$ that is the constant map. Hence, for $X = \{p\}$, the sequence

$$\cdots \xrightarrow{\partial_4} \mathcal{C}_3^{\rm sing}(X) \xrightarrow{\partial_3} \mathcal{C}_2^{\rm sing}(X) \xrightarrow{\partial_2} \mathcal{C}_1^{\rm sing}(X) \xrightarrow{\partial_1} \mathcal{C}_0^{\rm sing}(X) \to 0$$

amounts to a sequence of 1-dimensional spaces

$$\cdots \xrightarrow{\partial_4} \mathbb{R}[\sigma_3] \xrightarrow{\partial_3} \mathbb{R}[\sigma_2] \xrightarrow{\partial_2} \mathbb{R}[\sigma_1] \xrightarrow{\partial_1} \mathbb{R}[\sigma_0] \to 0$$

and, identifying $\mathbb{R}[\sigma_n]$ with $\mathbb{R}$:

$$\cdots \xrightarrow{\partial_4={\rm id}} \mathbb{R} \xrightarrow{\partial_3=0} \mathbb{R} \xrightarrow{\partial_2={\rm id}} \mathbb{R} \xrightarrow{\partial_1=0} \mathbb{R} \to 0$$

where "id" is the identity map, and hence the homology groups and Betti numbers work out the way we need. Note: if we did not fix a standard n-simplex Δ^n, but allowed singular homology to be built from maps of arbitrary simplices to X, then we would build our i-chains from all maps $S \to X$ where S is an arbitrary i-simplex; hence $\mathcal{C}_0(\{p\})$ would be built from the unique map $S \to \{p\}$ from an arbitrary 0-simplex, and we'd have to identify some of these or take some other step so that $H_0(\{p\})$ is one-dimensional.

(2) The above computation shows that the "degenerate simplicies" $\Delta^n \to X$ are essential to singular homology. Here is another example of the need for "degenerate simplicies:" if $\{u_0, u_1\}$ is a edge of K, then $[u_0, u_1] = -[u_1, u_0]$ in $\mathcal{C}_1^{\rm simp}(\mathsf{K})$; so it had better be true that in $H_1^{\rm sing}(|K|)$ we have $\xi(u_0, u_1) = -\xi(u_1, u_0)$. To see this, if (u_0, u_1, u_2) is a sequence of vertices of a simplex in K, with the u_i not necessarily distinct, then "barycentric coordinates" on (u_0, u_1, u_2) gives a map $\xi(u_0, u_1, u_2)\colon \Delta^2 \to |K|$ (whose image is either a 0-simplex, a 1-simplex, or a 2-simplex, depending on how many of u_0, u_1, u_2 are distinct). We easily see that — using the barycentric coordinates that determine $\xi(u_0, u_1, u_2)$ — that the restriction of $\xi(u_0, u_1, u_2)$ to its faces, and then standardizing these maps, are just maps $\Delta^1 \to X$ given by $\xi(u_1, u_2)$, $\xi(u_0, u_2)$, and $\xi(u_0, u_1)$; therefore

$$\partial_2 \xi(u_0, u_1, u_2) = \xi(u_1, u_2) - \xi(u_0, u_2) + \xi(u_0, u_1)$$

which is therefore 0 in $H_1^{\rm sing}(|K|)$. Taking $u_0 = u_1 = u_2$ shows that

$$\xi(u_0, u_0) - \xi(u_0, u_0) + \xi(u_0, u_0) = 0 \quad \text{in } H_1^{\rm sing}(|K|),$$

and hence $\xi(u_0, u_0) = 0$ in $H_1^{\rm sing}(|K|)$ (so $\xi(u_0, u_0)$ is therefore the constant map $\Delta^1 \to |K|$ taking each point in Δ^1 to u_0). Taking $u_0 = u_2 \neq u_1$ shows that $\xi(u_0, u_1) + \xi(u_1, u_0)$ is also 0 in $H_1^{\rm sing}(|K|)$. It follows that $\xi(u_0, u_1) = -\xi(u_1, u_0)$ in $H_1^{\rm sing}(|K|)$. So the use of degenerate maps $\Delta^2 \to |K|$ are needed to show that Ξ is well-defined, i.e., taking $[u_0, u_1] + [u_1, u_0]$ to 0 in $H_1^{\rm sing}(|K|)$.

(3) The basic approach to proving the isomorphism (51) is quite simple and works by induction: to prove (51) for a simplicial complex, K, imagine and that there $K_1, K_2 \subset K$ with $K_1 \cup K_2 = K$, and that (51) is an isomorphism for $K_1 \cap K_2, K_1, K_2$. Say that we have proved the Meyer-Vietoris theorem in singular homology (this is slightly trickier than in simplicial

homology). This means that we would have a diagram whose rows are long exact sequences:

$$\begin{tikzcd}
H_i(\mathsf{K}_1 \cap \mathsf{K}_2) \arrow[r] \arrow[d] & H_i(\mathsf{K}_1 \amalg \mathsf{K}_2) \arrow[r] \arrow[d] & H_i(\mathsf{K}) \arrow[r] \arrow[d, dashed] & H_{i-1}(\mathsf{K}_1 \cap \mathsf{K}_2) \arrow[r] \arrow[d] & H_{i-1}(\mathsf{K}_1 \amalg \mathsf{K}_2) \arrow[d] \\
H_i(|K_1| \cap |K_2|) \arrow[r] & H_i(|K_1| \amalg |K_2|) \arrow[r] & H_i(|K|) \arrow[r] & H_{i-1}(|K_1| \cap |K_2|) \arrow[r] & H_{i-1}(|K_1| \amalg |K_2|)
\end{tikzcd}$$

and where all vertical arrows except the middle ones are known to be isomorphisms. In the case the "five lemma" (another lemma in homological algebra) states that if the middle arrow exists (i.e., we have to show that $(u_0, \ldots, u_i) \mapsto \xi(u_0, \ldots, u_i)$ gives a well-defined map Ξ in (51)), then if the above diagram commutes, then the middle arrow is also an isomorphism (you can get by with a slightly weaker assumption on the first vertical arrow and on the last).

(4) The above is an oversimplification, because in general the Mayer-Vietoris sequence in singular homology for topological spaces $X_1, X_2 \subset X$ requires X to be the union of the *interiors* of X_1, X_2. Hence, when $X = |K|$ for a simplicial complex, K, if $K_1, K_2 \subset K$ are subcomplexes with $K_1 \cup K_2 = K$, we cannot take $X_i = |K_i|$ if we want to apply the usual Mayer-Vietoris sequence in singular homology; if we "thicken" X_1, X_2, by defining $\tilde{X}_i$ to be those points of distance $\le \epsilon$ to X_i in X (distance makes sense since K is a complex in $\mathbb{R}^N$), then we could show that $\tilde{X}_i$ has X_i as a deformation retract, and we could then apply the Mayer-Vietoris sequence to $\tilde{X}_i$, whose homology groups equal those of X_i (and similarly for $\tilde{X}_1 \cap \tilde{X}_2$ and $X_1 \cap X_2$). But more often one uses different long exact sequences akin to the Mayer-Vietoris sequence. For example, if $A \subset X$ are topological spaces, we define

$$\mathcal{C}_i^{\mathrm{sing}}(X, A) \stackrel{\mathrm{def}}{=} \mathcal{C}_i^{\mathrm{sing}}(X)/\mathcal{C}_i^{\mathrm{sing}}(A),$$

and defines the *relative homology groups* $H_i(X, A)$ to be the homology groups of the sequence (which we easily check is a chain complex):

$$\cdots \xrightarrow{\partial_3} \mathcal{C}_2^{\mathrm{sing}}(X, A) \xrightarrow{\partial_2} \mathcal{C}_1^{\mathrm{sing}}(X, A) \xrightarrow{\partial_1} \mathcal{C}_0^{\mathrm{sing}}(X, A) \to 0$$

The advantage is that it is immediate that

$$0 \to \mathcal{C}_i^{\mathrm{sing}}(A) \to \mathcal{C}_i^{\mathrm{sing}}(X) \to \mathcal{C}_i^{\mathrm{sing}}(X, A) \to 0$$

is short exact for all i, and this yields a long exact sequence

$$\cdots \to H_i^{\mathrm{sing}}(A) \to H_i^{\mathrm{sing}}(X) \to H_i^{\mathrm{sing}}(X, A) \to H_{i-1}^{\mathrm{sing}}(A) \to \cdots$$

The relative homology groups are often easy to determine. Moreover, in certain cases, in X/A, which is $X/\sim$ where $\sim$ identifies all points in A as a single point, then

$$\forall i \ge 1, \quad H_i^{\mathrm{sing}}(X, A) = H_i^{\mathrm{sing}}(X/A) \tag{52}$$

where X/A is $X/\sim$ where $\sim$ identified all points in A (which therefore becomes a single point in X/A).

(5) As a test of relative homology groups, let $A \subset X$ be the case where $A = X$. Then X/A is a single point, p. Clearly, in this case

$$\mathcal{C}_i^{\mathrm{sing}}(X, X) = \mathcal{C}_i^{\mathrm{sing}}(X)/\mathcal{C}_i^{\mathrm{sing}}(X) = 0,$$

so all the groups $H_i(X,X)$ vanish. Hence (52) holds, but it does not hold when $i=0$; to allow the condition $i=0$ in (52), we can use *reduced homology*, $\tilde H_i$, which simply reduces β_0 by 1 (one defines $\mathcal{C}_{-1}$ to be $\mathbb{R}$, and takes $\mathcal{C}_0 \to \mathcal{C}_{-1}$ to be the map that sums the scalar coefficients of a formal $\mathbb{R}$-linear sum). (For relative homology with $A\subset X$ and X connected, we still have $\tilde H_0(X,A)=0$, since we take $\mathcal{C}_{-1}(X,A)=\mathcal{C}_{-1}(X)/\mathcal{C}_{-1}(A)=\mathbb{R}/\mathbb{R}=0$.) The resulting homology groups are denoted with tildes, and (52) becomes

$$\forall i, \quad \tilde H_i^{\mathrm{sing}}(X,A)=\tilde H_i^{\mathrm{sing}}(X/A)$$

(for "good pairs" $A\subset X$). [Relatively homology is also good for suspensions, since we know that $\beta_1(\mathrm{Cone}_P(\mathsf{K}))=\beta_0(\mathsf{K})-1=\tilde\beta_0(\mathsf{K})$.]

Again, for more of the story above, you can start reading Hatcher's textbook [Hat02], starting at page 127 there, Section 2.1.

9.11. **Δ-Complexes.** Simplicial complexes are quite inefficient to describe certain topological spaces. Indeed, a one-dimensional simplicial complex gives rise to only *simple* graphs, meaning graphs without self-loops and multiple edges; in other words, at most one edge (or 1-simplex) can join any two vertices of a simplicial complex, and a 1-simplex isn't allowed to have the same two endpoints. In graph theory, it is often far more efficient to work graphs that can have multiple edges and self-loops. Δ-complexes are analogs of simplicial complexes where we allow (some possible) higher dimension analogs of multiple edges and self-loops.

Δ-complexes is a term introduced — or, at least, popularized — in Hatcher [Hat02] (Section 2.1). Let us limit ourselves to finite Δ-complexes here. We will borrow Hatcher's definitions in Section 2.1, slightly changing the notion; we refer the reader to Hatcher's textbook for more details.

Definition 9.53. Let X be a topological space. A Δ-*complex structure on* X is a collection $\{\sigma_\alpha\}_{\alpha\in A}$ of maps $\sigma_\alpha\colon\Delta^{n_\alpha}\to X$ where $n_\alpha\ge 0$ is an integer, such that

(1) if $\Delta^{n_\alpha}_\circ$ denotes the interior of Δ^{n_α}, then σ_α restricted to $\Delta^{n_\alpha}_\circ$ is injective;
(2) each element of X lies in $\sigma_\alpha(\Delta^{n_\alpha}_\circ)$ for a unique $\alpha\in A$;
(3) for any α and integer $0\le j\le n_\alpha$, for some $\beta\in A$ we have

$$\sigma_\beta=\mathrm{Stand}\bigl(\sigma|_{(\mathbf{e}_1,\ldots,\widehat{\mathbf{e}_j},\ldots,\mathbf{e}_{n_\alpha+1})}\bigr) \tag{53}$$

(4) a set $U\subset X$ is open iff for all $\alpha\in A$, $\sigma_\alpha^{-1}(U)\subset\Delta^{n_\alpha}$ is open (i.e., relatively open in Δ^{n_α} as a subset of $\mathbb{R}^{n_\alpha}$).

(Also we may as well assume that the $\{\sigma_\alpha\}_{\alpha\in A}$ are distinct maps, since we may as well discard any repeated maps.) The *i-simplicies of this Δ-complex* refers to all the $\sigma_\alpha\colon\Delta^{n_\alpha}\to X$ such that $n_\alpha=i$.

The above definition ensures us that X is isomorphic to

$$\left(\coprod_{\alpha\in A}\Delta^{n_\alpha}\right)\Big/\sim,$$

where $\amalg$ is the disjoint union, $\sim$ identifies the j-th $(n-1)$-dimensional boundary component of each Δ^{n_α} with Δ^{n_β} with $\beta\in A$ as in (53).

Example 9.54. Any simplicial complex comes with a canonical structure of a Δ-complex. Namely, if K is a simplicial complex, and $X = |K|$, then we orient each simplex $S \in K$, so that when we standardize the map $S \to |K| = X$ we get a map a map $\sigma_S \colon \Delta^{n_S} \to X$ where $n_S = \dim(S)$. We easily see check that $\{\sigma_S\}_{S\in K}$ gives X the structure of a Δ-complex.

Example 9.55. Δ-complexes are often a simpler way to describe topological spaces. For example, $\mathbb{S}^1$ can be described as the space

$$X = [0,1]/\{0,1\},$$

i.e., interval $[0,1]$, with 0 and 1 identified as a single point $p \in X$. Hence, $\mathbb{S}^1$ can be endowed with the structure of a Δ-complex $\{\sigma_\alpha, \sigma_\beta\}$, where $\sigma_\alpha \colon \Delta^1 \to X$ takes the interior of Δ^1 to $(0,1) \subset X$, and where σ_β is the map $\Delta^0 \to X$ taking 1 to $p = 0 = 1 \in X$. This realizes $\mathbb{S}^1$ as essentially one interval whose endpoints both equal a single point, which is a graph with one vertex and one self-loop. By contrast, if K is a simplicial complex with $|K|$ homeomorphic to $\mathbb{S}^1$, then as a graph, the abstract simplicial complex of K must have at least three vertices and three edges.

Example 9.56. The sphere $X = \mathbb{S}^n$ can be endowed with the structure of a Δ-complex as follows: we have two n-simplexes $\Delta^n \to X$, whose boundaries are glued together. Hence we can endow X with the structure of a Δ-complex with two n-simplices, and whose lower dimensional simplicies are the boundary of a single n-simplex.

Example 9.57. The spaces $\mathbb{T}^2$, $\mathbb{RP}^2$, and the Klein bottle, can be realized as $[0,1]^2/\sim$ where $\sim$ is the appropriate identitfication of points in the boundary of $[0,1]^2$. See Hatcher's textbook [Hat02], Section 2.1 (page 102). By drawing a diagonal edge, these can each be given a Δ-complex structure with only two 2-simplicies. (DRAW PICTURE IN CLASS.)

Like abstract simplicial complexes, if X is endowed with the structure of a Δ-complex, $\{\sigma_\alpha\}_{\alpha\in A}$, then X has a natural "simplicial homology:" namely, we let $\mathcal{C}_i^{\mathrm{simp}}(X)$ be the formal $\mathbb{R}$-linear combination of its i-dimensional simplicies, and for each i-simplex σ_α we define

$$\partial_i(\sigma_\alpha) = \sum_{j=1}^{i+1} (-1)^j \mathrm{Stand}\big(\sigma|_{(\mathbf{e}_1,\ldots,\widehat{\mathbf{e}_j},\ldots,\mathbf{e}_{i+1})}\big).$$

This gives rise to a chain

$$\cdots \xrightarrow{\partial_3} \mathcal{C}_2^{\mathrm{simp}}(X) \xrightarrow{\partial_2} \mathcal{C}_1^{\mathrm{simp}}(X) \xrightarrow{\partial_1} \mathcal{C}_0^{\mathrm{simp}}(X) \to 0,$$

and we define the *simplicial homology* of the Δ-complex structure on X to be the homology groups of this chain. Again, each σ_α can be viewed as a singular chain, which extends to a morphism

$$H_i^{\mathrm{simp}}(X) \to H_i^{\mathrm{sing}}(X),$$

which turns out to be an isomorphism (see Hatcher [Hat02], page 128).

9.12. **Singular Homology 4: Cubical Singular Homology.** Cubical singular homology is a bit simpler to work with when it comes to *homotopy* and is quite popular in introductory textbooks; otherwise it is defined similarly to simplicial singular homology. Here are the basics. (See [Mas80, Ful95, Wei14].)

[Simplicial singular homology has the advantage that it is more readily related to simplicial complexes and Δ-complexes.]

Let X be any topological space. Let $I = [0,1] \subset \mathbb{R}$; for $n \geq 0$ define the *standard n-cube* to be $I^n \subset \mathbb{R}^n$. By a *singular n-dimensional cube on X* we mean a map $I^n \to X$. The cube I^n has $2n$ "faces," each being a subcube of dimension $(n-1)$: so to any singular cube $f\colon I^n \to X$ and $1 \leq j \leq n$, the *front j-th side* is the map $I^{n-1} \to X$:

$$\mathrm{Front}_j(f) \stackrel{\mathrm{def}}{=} f(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n),$$

and similarly its *back j-th side*

$$\mathrm{Back}_j(f) \stackrel{\mathrm{def}}{=} f(x_1, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_n)$$

(i.e., $x_j = 0$ is the "front," and $x_j = 1$ is the "back" here). We define the *boundary of f* to be the formal combination

$$\partial_n f \stackrel{\mathrm{def}}{=} \sum_{j=1}^{n} (-1)^j \bigl(\mathrm{Front}_j(f) - \mathrm{Back}_j(f)\bigr).$$

[For example, for $n = 1$ we have $\partial_1 f = f(1) - f(0)$ (a formal difference of two constant functions on I^0), which should make sense if $[0,1]$ is oriented "from 0 to 1 in $\mathbb{R}$." For $n = 2$,

$$\partial_2 f = f(1, x_1) - f(0, x_1) + f(0, x_1) - f(1, x_1),$$

and formal $\mathbb{R}$-linear sum of 4 functions, which is what you'd expect if $I^2 \subset \mathbb{R}^2$ is given the usual orientation of $dx_1 \wedge dx_2$, i.e., counterclockwise.]

We easily see that $\partial_{n-1}\partial_n = 0$. We then define $\mathcal{Q}_i^{\mathrm{sing(cube)}}(X)$ to be the $\mathbb{R}$-linear combination of singular i-cubes on X, and we have a chain of vector spaces

$$\cdots \xrightarrow{\partial_3} \mathcal{Q}_2^{\mathrm{sing(cube)}}(X) \xrightarrow{\partial_2} \mathcal{Q}_1^{\mathrm{sing(cube)}}(X) \xrightarrow{\partial_1} \mathcal{Q}_0^{\mathrm{sing(cube)}}(X) \to 0, \tag{54}$$

and we are tempted to define the i-th homology group of this chain as the i-th (cubical) singular homology group.

This approach turns out to fail — quite spectacularly — when $X = \{p\}$ is a single point: for then (54) has each $\mathcal{Q}_n(\{p\})$ one-dimensional (for the unique constant map $I^n \to \{p\}$), and (56) becomes

$$\cdots \xrightarrow{\partial_3 = 0} \mathbb{R} \xrightarrow{\partial_2 = 0} \mathbb{R} \xrightarrow{\partial_1 = 0} \mathbb{R} \to 0, \tag{55}$$

whose i-th homology group is $\mathbb{R}$ for all i. To fix this, we essentially ignore (!) all the $\mathcal{Q}_i(\{p\})$ for $i \geq 1$: more precisely we say that a map $f\colon I^n \to X$ is *degenerate* if there is at least one coordinate that f is independent of, i.e., for some $1 \leq j \leq n$, $f = f(x_1, \ldots, x_n)$ is independent of j. We easily check that if f is degenerate, then $\partial_n f$ is a formal linear combination of degenerate functions $I^{n-1} \to X$ (the point is that $\mathrm{Front}_j - \mathrm{Back}_j$ vanishes on any function that is independent of x_j, and the remaining terms of $\partial_n f$ are degenerate functions). It follows that setting $\mathcal{D}_n(X)$ to be the formal linear combinations of degenerate maps $I^n \to X$, we have

$$\mathcal{D}_n(X) \subset \mathcal{Q}_n(X), \quad \partial_n\bigl(\mathcal{D}_n(X)\bigr) \subset \mathcal{D}_{n-1}(X),$$

and therefore ∂_n extends to a quotient map

$$\partial_n \colon \mathcal{C}_n(X) \to \mathcal{C}_{n-1}(X), \quad \text{where } \mathcal{C}_n(X) \stackrel{\text{def}}{=} \mathcal{Q}_n(X)/\mathcal{D}_n(X).$$

We call $\mathcal{C}_n(X)$ the *(cubical) singular n-chains of X*. Since $\partial_{n-1}\partial_n = 0$ on $\mathcal{Q}_n(X)$, the same holds on $\mathcal{C}_n(X)$, and we get a chain of vector spaces:

$$\cdots \xrightarrow{\partial_3} \mathcal{C}_2^{\text{sing(cube)}}(X) \xrightarrow{\partial_2} \mathcal{C}_1^{\text{sing(cube)}}(X) \xrightarrow{\partial_1} \mathcal{C}_0^{\text{sing(cube)}}(X) \to 0. \tag{56}$$

We define the *(cubical) singular homology groups* of X to be the homology groups of the above chain, whose i-th group is denoted $H_i^{\text{sing(cube)}}(X)$, and we define the *(cubical) singular Betti numbers of X* to be, as expected

$$\beta_i^{\text{sing(cube)}}(X) \stackrel{\text{def}}{=} \dim\bigl(H_i^{\text{sing(cube)}}(X)\bigr).$$

Remark 9.58. Because degenerate functions are crucial to simplicial singular homology, the above suggested fix for cubical singular homology — i.e., passing from $\mathcal{Q}_i(X)$ to $\mathcal{C}_i(X)$ — may seem intuitively questionable. To develop some intuition as to why quotienting by $\mathcal{D}_i(X) \subset \mathcal{Q}_i(X)$ "works," first note: $H_0(X)$ is the same whether we work with $\mathcal{Q}_i(X)$ or $\mathcal{C}_i(X)$. However, $H_1(\{p\})$ is different; moreover, the constant map $I \to \{p\}$ needs be 0 in $H_1(\{p\})$, and cubical singular homology has the "defect" that $I \to \{p\}$ is not in the image of ∂_2. By contrast, in simplicial singular homology the map $\Delta^2 \to \{p\}$ plays the crucial role of having ∂_2 take this map to $\Delta^1 \to \{p\}$, and the constant map $\Delta^1 \to \{p\}$ is the analog of the cubical singular map $I \to \{p\}$. Hence quotienting out by $\mathcal{D}_i(\{p\})$ is needed to correct this "defect" of the chains $\mathcal{Q}_i(\{p\})$ regarding $H_1(\{p\})$. Another possible correction would be to add to $\mathcal{C}_2(\{p\})$ some element(s) whose image(s) via ∂_2 would be the constant map $I \to \{p\}$.

Remark 9.59. Note that in simplicial singular homology, the map $\Delta^n \to \{p\}$ vanishes in $H^n(\{p\})$ iff n is odd; indeed, the map $\Delta^0 \to \{p\}$ does not vanish, and gives rise to the one-dimensional vector space $H_0^{\text{sing(simp)}}(\{p\})$. The map $\Delta^2 \to \{p\}$ does not appear in $H_2^{\text{sing(simp)}}(\{p\})$, simply because $\Delta^2 \to \{p\}$ is not in the kernel of ∂_2. But if one defined the constant map $\Delta^2 \to \{p\}$ as 0 in the chain $\mathcal{C}_2^{\text{sing(simp)}}(X)$, the groups $H_i^{\text{sing(simp)}}(\{p\})$ would be incorrect.

Remark 9.60. Many topology textbooks define the *fundamental group*, $\pi_1(X)$, of a topological space, X, before defining homology groups. The fundamental group is built by fixing a point $x_0 \in X$ (assuming X is connected), and $\pi_1(X)$ is defined as the set of paths $p\colon [0,1] \to X$ with $p(0) = p(1) = x_0$, and where the constant path (i.e., $p(t) = x_0$ for all $t \in [0,1]$) plays the role of the identity. $H_1(X)$ is essentially the abelianization of $\pi_1(X)$ (in our case tensored with $\mathbb{R}$, since we have defined $H_1(X)$ as a $\mathbb{R}$-vector space). This is another reason why the constant path should be 0 in $H_1(X)$ (although this doesn't directly motivate the formulation of $\mathcal{C}_i = \mathcal{Q}_i/\mathcal{D}_i$).

9.13. **Singular Homology 5: Homotopy Equivalence.** TO BE CONTINUED...

10. Barcodes for Simplicial Complexes

10.1. **Point Clouds and Simplicial Complexes.** In class, March 26, 2025, we explained how point clouds give rise to an increasing sequence of abstract simplicial complexes. This gives rise to a sequence

$$\mathsf{K}^0 \xrightarrow{f^0} \mathsf{K}^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{n-1}} \mathsf{K}^n,$$

where the $f_i \colon \mathsf{K}^i \to \mathsf{K}^{i+1}$ are inclusions. The discussion in this section applies, more generally, when K^i are arbitrary simplicial complexes, and $f^i \colon \mathsf{K}^i \to \mathsf{K}^{i+1}$ are arbitrary maps of simplicial complexes (i.e., f^i maps the vertices of K^i to those of K^{i+1}, in a way that each k-simplex of K^i is taken to a k-simplex of K^{i+1}. As we explained in class, for each $j = 0, 1, 2, \ldots$ we get a sequence of maps:

$$H_j(\mathsf{K}^0) \xrightarrow{f^0_*} H_j(\mathsf{K}^1) \xrightarrow{f^1_*} \cdots \xrightarrow{f^{n-1}{}_*} H_j(\mathsf{K}^n)$$

(of course, the $f^i_* = f^i_{*,j}$ depend on j, but it is customary to drop the j, which is clear from the context).

Similarly, if

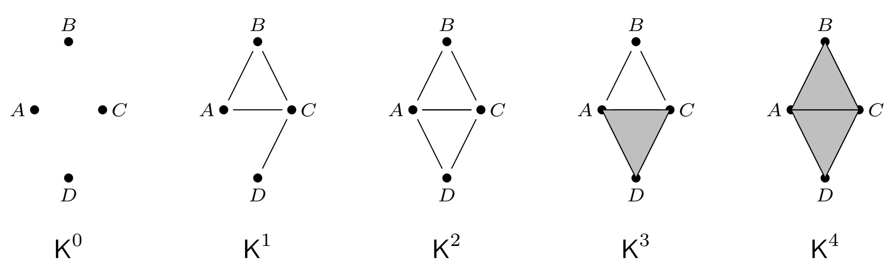
$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{n-1}} X^n$$

is a sequence of continuous maps of topological spaces, then each f^i maps each singular j-simplex of X^i to one of X^{i+1} (by composing with f^i), and we similarly get a map for each j:

$$H_j(X^0) \xrightarrow{f^0_*} H_j(X^1) \xrightarrow{f^1_*} \cdots \xrightarrow{f^{n-1}{}_*} H_j(X^n)$$

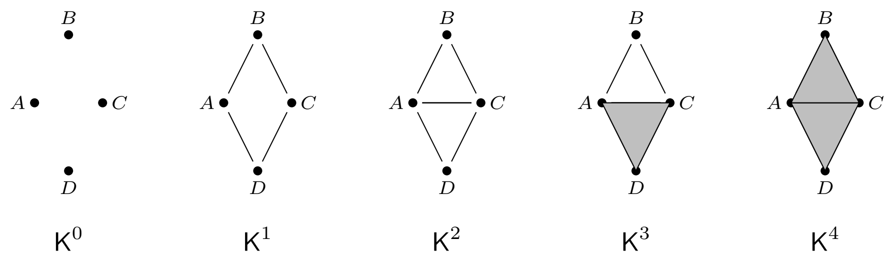
On March 26, 2025 we drew some examples and asked ourselves: is there a "most persistent element of $H_1(\mathsf{K}^i)$?

Example 10.1. Consider the sequence of complexes:



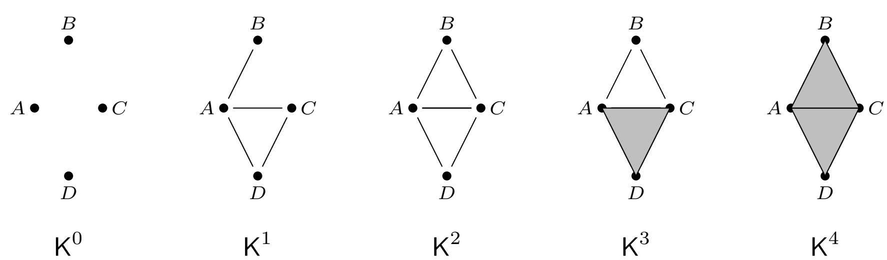
(these should look familiar from Example 4.9). The associated sequence of zeroth Betti numbers, which reflects the number of connected components, is not particularly interesting here. Then the sequence of first Betti numbers, $\beta_1(\mathsf{K}^j) = \dim(H_1(\mathsf{K}^j))$ is $0, 1, 2, 1, 0$. Notice that the cycle $\tau_1 = [A, B] + [B, C] + [C, A]$, which first appears in K^1, remains a cycle in $H_i(\mathsf{K}^j)$ for $j = 2, 3, 4$ and is non-zero in $H_1(\mathsf{K}^2)$ and $H_1(\mathsf{K}^3)$. Hence there is an obvious sense in which τ_1 is the "most persistent H_1 feature" of this sequence.

Example 10.2. Consider the same sequence of complexes with a slightly different K^1:



The cycle $\tau_1 = [A,B]+[B,C]+[C,A]$ doesn't appear until K^2, but the cycle $\tau_2 = [A,B]+[B,C]+[C,D]+[D,A]$ appears in K^1 and is nonzero in homology there and in K^2 and K^3. Hence τ_2 is the "most persistent H_1 feature" of this sequence.

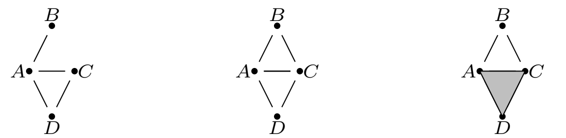
Example 10.3. Let us again modify K^1:

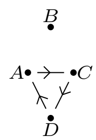
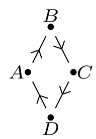


The cycle $\tau_3 = [A,C]+[C,D]+[D,A]$ appears in K^1, but vanishes in $H_1(\mathsf{K}^3)$. The cycles $\tau_1 = [A,B]+[B,C]+[C,A]$ and $\tau_2 = [A,B]+[B,C]+[C,D]+[D,A]$ don't appear until K^2, but remain nonzero in homology in K^3. So there is "fully persistent" cycle in this example.

Now the question is can one describe the homology and "persistent cycles" in some simplified diagram. Let us give some examples of what we have in mind.

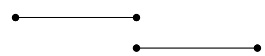
Example 10.4. Consider Example 10.3 and how $H_1(\mathsf{K}^i)$ "evolves in i:" since $H_1(\mathsf{K}^i) = 0$ for $i = 0, 4$, we will focus on $i = 1, 2, 3$:



	$H_1(\mathsf{K}^1) \simeq \mathbb{R}$	$H_1(\mathsf{K}^2) \simeq \mathbb{R}^2$	$H_1(\mathsf{K}^3) \simeq \mathbb{R}$
$\tau_3 =$ 	$\mathbb{R}$	$\mathbb{R}$	0
$\tau_2 =$ 	0	$\mathbb{R}$	$\mathbb{R}$

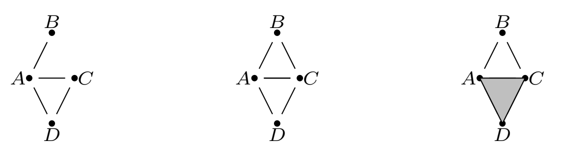
The meaning of this diagram is that τ_3 exists in K^1 and is nonzero in H_1 for $\mathsf{K}^1, \mathsf{K}^2$, but then is 0 in $H_1(\mathsf{K}^3)$. Similarly, τ_2 does not exist in K^1, but does in $\mathsf{K}^2, \mathsf{K}^3$ and is nonzero in H_1 there. Moreover, τ_3, τ_2 form a basis for $H_1(\mathsf{K}^2)$. Hence we can represent $H_1(\mathsf{K}^i)$ for $i = 1, 2, 3$ in simpler terms:

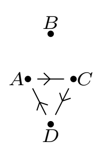
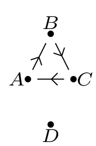
τ_3
τ_2



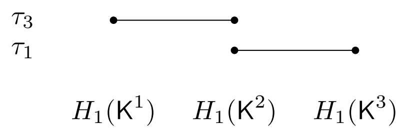
$H_1(\mathsf{K}^1)$ $H_1(\mathsf{K}^2)$ $H_1(\mathsf{K}^3)$

which is called a "barcode diagram." Note that there is some freedom here: one could alternatively replace τ_2 with τ_1 above, yielding the chart:



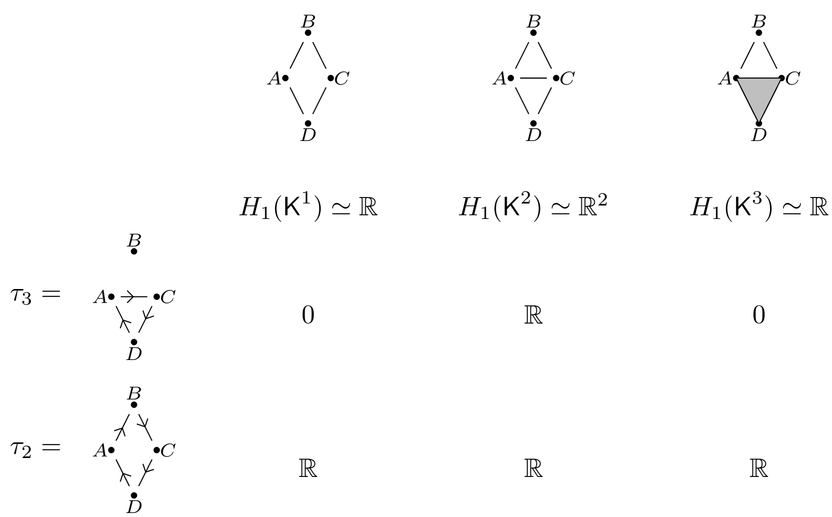
	$H_1(\mathsf{K}^1) \simeq \mathbb{R}$	$H_1(\mathsf{K}^2) \simeq \mathbb{R}^2$	$H_1(\mathsf{K}^3) \simeq \mathbb{R}$
$\tau_3 =$ 	$\mathbb{R}$	$\mathbb{R}$	0
$\tau_1 =$ 	0	$\mathbb{R}$	$\mathbb{R}$

which yields the same "barcode diagram"

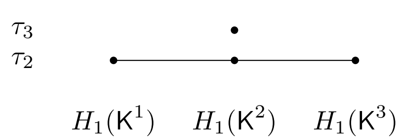


The point is that τ_1 and τ_2 both first appear in K^2, and τ_1, τ_3 form another basis for $H_1(\mathsf{K}^2)$, and $\tau_1 = \tau_2$ as elements of $H_1(\mathsf{K}^3)$.

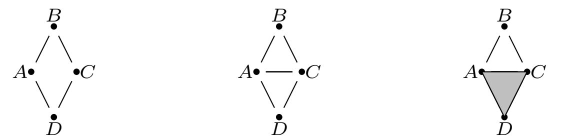
Example 10.5. One can create a similar "simplification" of H_1 for Example 10.2:

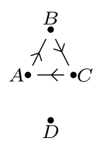
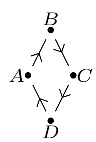


which yields the "homology simplification" or "barcode" for H_1:



Example 10.6. Warning: in the last example, we cannot replace τ_3 with τ_1, since individually the picture now looks like:



	$H_1(\mathsf{K}^1) \simeq \mathbb{R}$	$H_1(\mathsf{K}^2) \simeq \mathbb{R}^2$	$H_1(\mathsf{K}^3) \simeq \mathbb{R}$
$\tau_1 =$ 	0	$\mathbb{R}$	$\mathbb{R}$ ☹
$\tau_2 =$ 	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$

The problem is that although τ_1, τ_2 form a basis for $H_1(\mathsf{K}^2)$, and both are nonzero in $H_1(\mathsf{K}^3)$, they are linearly dependent in the one-dimensional space $H_1(\mathsf{K}^3) \simeq \mathbb{R}$. So we want to identify a set of 1-forms, each that is first found in some $H_1(\mathsf{K}^i)$, but the set of 1-forms that are non-zero in each $H_1(\mathsf{K}^j)$ are linearly independent there. In this example, τ_2 is the unique 1-form (up to scalar multiplication) that persists for the entire sequence; hence we must use τ_2 in such a "simplification" or "barcode." However, the second 1-form we choose in K^2 needs to be taken to 0 in K^3.

The existence of an appropriate basis of 1-forms to make a simplification or barcode with the right independence properties can be done in a very general context, which we next describe as the "abstract barcode" theorem.

10.2. **The Abstract Barcode Theorem.** The "barcode theorem" is a theorem in linear algebra that is an integral part of persistent homology, first discovered in [ELZ02, ELZ00]. Yet, the "barcode theorem" can be viewed as a general theorem in linear algebra, and specifically [CZCG04] a consequence of the structure of graded modules over a PID.

We will describe a few simple algorithms to find "bar codes." Let us give the relevant definitions and results.

Throughout this article we work with the field of scalars $\mathbb{R}$, although the same discussion holds over an arbitrary field.

Definition 10.7. Let $n \geq 0$ be an integer. A string of $\mathbb{R}$-vector spaces of length $n+1$ refers to the data consisting of a sequence $V^0, \ldots, V^n$ of $\mathbb{R}$-vector spaces, and linear maps $\mathcal{L}^i \colon V^i \to V^{i+1}$ for $i = 0, \ldots, n-1$. We often use the symbols $V^\cdot, \mathcal{L}^\cdot$ to refer collectively to $\{V^i\}_{0 \leq i \leq n}$ and $\{\mathcal{L}^i\}_{0 \leq i \leq n-1}$, and $(V^\cdot, \mathcal{L}^\cdot)$ to the string.

We may represent a string of vector spaces with the diagram:

$$V^0 \xrightarrow{\mathcal{L}^0} V^1 \xrightarrow{\mathcal{L}^1} \cdots \xrightarrow{\mathcal{L}^{n-1}} V^n. \tag{57}$$

Definition 10.8. In Definition 10.7, let i, j be integers with $0 \le i \le j \le n$. For $0 \le k \le n$, let $V_k = \mathbb{R}$ if $i \le k \le j$, and otherwise let $V_k = 0$. For $i \le k \le j-1$, let $\mathcal{L}^k$ be the identity map. We call this string of $n+1$ $\mathbb{R}$-vector spaces the *(i,j)-bar*. As a diagram, the (i,j)-bar can be represented as:

$$0 \to \cdots \to 0 \to \mathbb{R} \to \cdots \to \mathbb{R} \to 0 \to \cdots \to 0,$$

where the first appearance of $\mathbb{R}$ is in V^i, and the last in V^j, and all morphisms $\mathbb{R} \to \mathbb{R}$ are the identity maps (there is only one morphism $0 \to \mathbb{R}$, and only one $\mathbb{R} \to 0$). By a *bar* we mean any (i,j)-bar.

Our main theorem states that any string of finite dimensional vector spaces is *isomorphic* to a *direct sum* of bars of the form in Definition 10.7. Hence we need to define the *direct sum*.

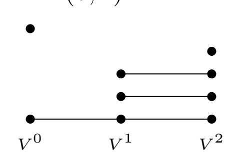
Definition 10.9. Let $\mathcal{S}_1 = (V_1^{\cdot}, \mathcal{L}_1^{\cdot})$, $\mathcal{S}_2 = (V_2^{\cdot}, \mathcal{L}_2^{\cdot})$ be strings of $\mathbb{R}$-vector spaces of the same length $n+1$. The *direct sum* of $\mathcal{S}_1$ and $\mathcal{S}_2$ is the string whose i-th vector space is $V_1^i \oplus V_2^i$, and whose i-th morphism is $\mathcal{L}_1^i \oplus \mathcal{L}_2^i$.

Hence the diagram representing the direct sum is:

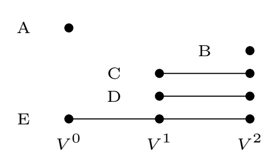
$$V_1^0 \oplus V_2^0 \xrightarrow{\mathcal{L}_1^0 \oplus \mathcal{L}_2^0} \cdots \xrightarrow{\mathcal{L}_1^{n-1} \oplus \mathcal{L}_2^{n-1}} V_1^n \oplus V_2^n.$$

The direct sum of any set of strings is similarly defined.

Example 10.10. Let $n = 2$. The direct sum of the bar $(0,0)$, the bar $(2,2)$, 2 copies of the bar $(1,2)$, and the bar $(0,2)$ is visualized by the *barcode*



To formalize this, we label each bar with a unique letter from $A, \ldots, E$:



This describes V^0, V^1, V^2 as

$$V^0 = \mathbb{R}^{\{A,E\}} \to V^1 = \mathbb{R}^{\{C,D,E\}} \to V^2 = \mathbb{R}^{\{B,C,D,E\}},$$

where we understand the following convention: if S is a set, then $\mathbb{R}^S$ (as usual) refers to the $\mathbb{R}$-vector space of maps $S \to \mathbb{R}$; if S_1, S_2 are two sets (one usually thinks of $S_1, S_2 \subset T$ as subsets of an "ambient" set T), then one defines a "canonical map" $\mathcal{K}_{S_1 \to S_2} \colon \mathbb{R}^{S_1} \to \mathbb{R}^{S_2}$ taking $\mathbf{v} \in \mathbb{R}^{S_1}$ to the function that agrees on $\mathbf{v}$ on $S_1 \cap S_2$, and otherwise, i.e., on $S_2 \setminus S_1$, takes the value 0.

We now want to formalize the notion of *isomorphic* strings of vector spaces.

Definition 10.11. Let $\mathcal{S}_1 = (V_1^{\cdot}, \mathcal{L}_1^{\cdot})$, $\mathcal{S}_2 = (V_2^{\cdot}, \mathcal{L}_2^{\cdot})$ be strings of $\mathbb{R}$-vector spaces of the same length $n+1$. A *morphism* $\mathcal{S}_1 \to \mathcal{S}_2$ is a collection of maps $\mathcal{M}^i \colon V_1^i \to V_2^i$ that intertwine with the morphisms of $\mathcal{S}_1$ and $\mathcal{S}_2$ in the evident sense, i.e., for all $0 \le i \le n$, we have $\mathcal{M}^{i+1}\mathcal{L}_1^{i+1} = \mathcal{L}_2^i \mathcal{M}^i$ for all i.

Hence we can depict this morphism with a "commutative diagram":

$$\begin{tikzcd}
V_1^0 \arrow[r, "\mathcal{L}_1^0"] \arrow[d, "\mathcal{M}^0"] & V_1^1 \arrow[r, "\mathcal{L}_1^1"] \arrow[d, "\mathcal{M}^1"] & \cdots \arrow[r, "\mathcal{L}_1^{n-2}"] & V_1^{n-1} \arrow[r, "\mathcal{L}_1^{n-1}"] \arrow[d, "\mathcal{M}^{n-1}"] & V_2^n \arrow[d, "\mathcal{M}^n"] \\
V_2^0 \arrow[r, "\mathcal{L}_2^0"'] & V_2^1 \arrow[r, "\mathcal{L}_2^1"'] & \cdots \arrow[r, "\mathcal{L}_2^{n-2}"'] & V_2^{n-1} \arrow[r, "\mathcal{L}_2^{n-1}"'] & \bullet
\end{tikzcd}$$

It is immediate that this morphism is an isomorphism (i.e., this morphism has an inverse morphism) iff each $\mathcal{M}^i$ is an isomorphism.

The main point of this subsection is to prove the following theorem, and to give an algorithm in the general case.

Theorem 10.12. *Any string, $\mathcal{F}$, of length $n+1$ of finite dimensional vector spaces is isomorphic to a direct sum of bars. Moreover, for each $0 \le i \le j \le n$, the number of (i,j)-bars in this direct sum is independent of this direct sum.*

Definition 10.13. The direct sum of bars for a string, $\mathcal{F}$, as in Theorem 10.12 is called a *barcode decomposition of $\mathcal{F}$*.

We will now give different algorithms for finding these bars, all of which provide a proof of Theorem 10.12. Let us describe these in rough terms:

(1) To any string, $\mathcal{F}$, there is a natural "total space" of $\mathcal{F}$, namely the direct sum of the vector spaces of $\mathcal{F}$, and a linear map on this "total space" that combines the linear maps in $\mathcal{F}$. This is easily seen to be nilpotent; then the usual algorithm for bringing a nilpotent matrix into Jordan canonical can be adapted to provide a barcode decomposition for $\mathcal{F}$.

(2) There is a simple "forward sweeping" algorithm that finds all the bars beginning at V^0, then those at V^1, etc. The idea is that all the bars beginning in V^0 span the the "sub-string" of $\mathcal{F}$ consisting of the images of V^0 in the V^i for $i \ge 1$, i.e.,

$$V^0 \xrightarrow{\mathcal{L}^0} V^{1,0} = \mathcal{L}^0(V^0) \xrightarrow{\mathcal{L}^1} \cdots \xrightarrow{\mathcal{L}^{n-1}} V^{n,0} = \mathcal{L}^{n-1}\mathcal{L}^{n-2}\ldots\mathcal{L}^0(V^0); \tag{58}$$

furthermore, the subspaces $V^{i,0} \subset V^i$ are fixed, regardless of how any of the bars for $\mathcal{F}$ are chosen. Hence the choice of bars beginning in V^1, V^2, etc., is independent of our choice for bars beginning in V^0. So we first find a set of bars for (58) (which we can do by longest bar first). We then find bars beginning in V^1 which in V^1 give a basis for $V^1/V^{1,0}$. We then find bars for V^i for $i = 2, 3, \ldots, n$ in this way.

(3) There are likely many variants of the above algorithms. See Remark 10.19 regarding the algorithm in the textbook by Horn and Johnson for finding Jordan canonical form starting with a Schur decomposition.

10.3. **The Forward Sweep Algorithm.** In this subsection we prove Theorem 10.12 using what we call the *forward sweep* algorithm. We will introduce a lot of terminology and notation without formally writing out definitions.

What we will discover, after explaining the algorithm, is that although our "forward sweep" algorithm requires some $\binom{n+2}{2}$ "sweeps" which we gather into $n+1$ phases. However, it will turn out that each of these sweeps can be performed "independently," using information determined by the string but independent of the choice of bars in each phase of the algorithm.

Consider a string $\mathcal{F} = (V^\cdot, \mathcal{L}^\cdot)$ of vector spaces:

$$V^0 \xrightarrow{\mathcal{L}^0} V^1 \xrightarrow{\mathcal{L}^1} \cdots \xrightarrow{\mathcal{L}^{n-1}} V^n.$$

For $0 \le i \le j \le n$ we let $\mathcal{L}^{i\to j} \colon V^i \to V^j$ be the evident composition, i.e.,

$$\mathcal{L}^{i\to j} = \mathcal{L}^{j-1} \circ \mathcal{L}^{j-2} \circ \ldots \circ \mathcal{L}^i \colon V^i \to V^j$$

(understanding $\mathcal{L}^{i\to i}$ to be the identity map), and we let

$$V^{i\to j} = \mathrm{Image}(\mathcal{L}^{i\to j}) = \mathcal{L}^{i\to j}(V^i) \subset V^j$$

(understanding $V^{i\to i} = V^i$), and

$$m_{i,j} = \dim\bigl(V^{i\to j}\bigr), \quad m_i = m^{i,i} = \dim(V^i).$$

The V^0 *generated string of* $\mathcal{F}$ refers to the sub-string (a sub-string of $\mathcal{F}$ in the evident sense):

$$V^0 \to V^{0\to 1} \to \cdots \to V^{0\to n} \tag{59}$$

where the map $V^{0\to i} \to V^{0\to i+1}$ is the restrictions of $\mathcal{L}^i$. It will be crucial to note that all the maps in (59) are surjective, since for all $0 \le i \le n-1$,

$$V^{0\to i+1} = \mathcal{L}^i \mathcal{L}^{0\to i} V^0 = \mathcal{L}^i\bigl(V^{0,i}\bigr).$$

It follows that for any $i \le i'$, the map $V^{0\to i} \to V^{0\to i'}$, which is restriction of $\mathcal{L}^{i\to i'}$ to $V^{0\to i}$, is a composition of surjective maps, and is therefore surjective.

Our algorithm is based on the following observations: assume that a barcode decomposition of $\mathcal{F}$ exists; then bars that are nonzero in V^0 must contain a basis V^0, and the images of these vectors in V^i that are non-zero must be a basis for $V^{i,0}$. Hence, even if the choice of bars beginning at V^0 is not unique (and it generally is not), their spans in each V^i must equal $V^{0\to i}$. In this way if we produce a "full" set of $(0,q)$-bars for all q with $0 \le q \le n$, meaning bars whose direct sum equals (59), the selection of the (ℓ,q)-bars for $1 \le \ell \le q$ only depends on (59), and not the particular choice of bars. This observation will be crucial to what follows.

We will call the selection of all $(0,q)$-bars with $0 \le q \ne n$ the "0-th phase" of the algorithm.

10.3.1. *Phase* 0*, First Step: A Forward Sweep on the Longest Bars in* (59)*.* First we choose the $(0,n)$-bars, which represent the "longest" or "most persistent" bars of the string $\mathcal{F} = (V^\cdot, \mathcal{L}^\cdot)$.

Since $\dim(V^{0\to n}) = m_{0,n}$, we may choose a basis $w_i^{0,n}$, with $1 \le i \le m_{0,n}$ of $V^{0\to n}$ (which is the emptyset if $V^{0\to n} = 0$ and therefore $m_{0,n} = 0$). Since the map $V^0 \to V^{0\to n}$ is surjective, for each $1 \le i \le m_{0,n}$ we may choose $v_i^{0,n}$ such that $\mathcal{L}^{0\to n} v_i^{0,n}$. Each $v^{0,i}$ therefore determines a $(0,n)$-bar in $\mathcal{F} = (V^\cdot, \mathcal{L}^\cdot)$, namely

$$v^{0,i} \mapsto \mathcal{L}^{0\to 1} v^{0,i} \mapsto \mathcal{L}^{0\to 2} v^{0,i} \mapsto \cdots \mapsto \mathcal{L}^{0\to n} v^{0,i} = w^{0,i} \ne 0. \tag{60}$$

The number of these $(0,n)$-bars is $m_{0,n}$, and since

$$m_{0,n} = \dim\bigl(V^{0\to n}\bigr),$$

the number of these bars is depends only on $\mathcal{F} = (V^{\cdot}, \mathcal{L}^{\cdot})$. (Of course, the choice of the $w^{0,i}$ is not unique if $m_{0,n} \ge 1$.)

Now we claim that for any $0 \le j \le n$, the V^j part of the $(0,n)$-bars in (60) are linearly independent, i.e., the vectors $\mathcal{L}^{0\to j} v_i^{0,n}$ with $1 \le i \le m_{0,n}$ are linearly independent: indeed, say that for scalars $\alpha_1, \ldots, \alpha_{m_{0,n}}$ we have

$$\sum_{i=1}^{m_{0,n}} \alpha_i \mathcal{L}^{0,j} v_i^{0,n} = 0. \tag{61}$$

Then apply $\mathcal{L}^{j\to n}$ to both sides: since $\mathcal{L}^{j\to n}\mathcal{L}^{0\to j} v_i^{0,n} = \mathcal{L}^{0\to n} v^{0,i} = w_i^{0,n}$, applying $\mathcal{L}^{j\to n}$ to (61) gives

$$\sum_{i=1}^{m_{0,n}} \alpha_i w_i^{0,n} = 0.$$

But since the $w_i^{0,n}$ form a basis for $V^{0\to n}$, we must have $\alpha_i = 0$ for all i.

10.3.2. *Phase 0, Second Step: A Forward Sweep on the Second Longest Bars in* (59). The next step is to find all the $(0,n-1)$-bars, which need to be of the form

$$v^0 \mapsto \mathcal{L}^{0\to 1} v^0 \mapsto \ldots \mapsto \mathcal{L}^{0,n-1} v^0 \mapsto \mathcal{L}^{0,n} v^0 = 0$$

for some $v^0 \in V^0$. So for each $0 \le \ell \le q \le n-1$, let

$$K_{\ell,q} = \ker(\mathcal{L}^q) \cap V^{\ell\to q} = \{w \in V^{\ell\to q} \mid \mathcal{L}^q w = 0\}. \tag{62}$$

By the surjectivity of the maps in (59),

$$\dim(K_{0,n-1}) = \dim\bigl(V^{0\to n-1}\bigr) - \dim\bigl(V^{0\to n}\bigr) = m_{0,n-1} - m_{0,n}.$$

Choose a basis, $w_i^{0,n-1}$, with $1 \le i \le m_{0,n-1} - m_{0,n}$, for $K_{0,n-1} \subset V^{0\to n-1}$, and (by surjectivity) choose for each i a vector $v_i^{0,n-1} \in V^0$ such that $\mathcal{L}^{0\to n-1}(v_i^{0,n-1}) = w_i^{0,n-1}$. This gives for each $1 \le i \le m_{0,n-1} - m_{0,n}$ a $(0,n-1)$-bar

$$v_i^{0,n-1} \mapsto \mathcal{L}^{0\to 1} v_i^{0,n-1} \mapsto \ldots \mapsto \mathcal{L}^{0\to n-1} v_i^{0,n-1} \mapsto \mathcal{L}^{0\to n} v_i^{0,n-1} = 0.$$

We now claim that for any j with $0 \le j \le n-1$, the images of the vectors

$$\{v_i^{0,n}\}_{i\in[m_{0,n}]} \cup \{v_i^{0,n-1}\}_{i\in[m_{0,n-1}-m_{0,n}]}$$

in V^j are linearly independent. The argument is similar to the preceding linear independence algorithm: namely, say that

$$\sum_{i=1}^{m_{0,n}} \alpha_i \mathcal{L}^{0,j} v_i^{0,n} + \sum_{i'=1}^{m_{0,n}} \beta_{i'} \mathcal{L}^{0,j} v_{i'}^{0,n} = 0. \tag{63}$$

Applying $\mathcal{L}^{j\to n}$ to both sides we get

$$\sum_{i=1}^{m_{0,n}} \alpha_i w_i^{0,n} = 0,$$

and hence all the α_i are zero; then applying $\mathcal{L}^{j\to n-1}$ we then get

$$\sum_{i'=1}^{m_{0,n}} \beta_{i'} w_{i'}^{0,n} = 0,$$

and therefore all the $\beta_{i'}$ are zero.

Similarly to before, the number of $(0,n-1)$-bars built above is $m_{0,n-1}-m_{0,n}$, which depends only on $\mathcal{F}=(V^{\cdot},\mathcal{L}^{\cdot})$.

10.3.3. *Phase 0: The General Forward Sweep Step for* (59). We similarly find the $(0,q)$-bars for any $0\le q\le n-1$: We choose a basis $w_i^{0,q}$ for $K_{0,q}$ as in (62), hence for $1\le i\le m_{0,q}-m_{0,q+1}$, and choose $v_i^{0,q}\in V^0$ such that $\mathcal{L}^{0\to q}v_i^{0,q}=w_i^{0,q}$. An argument similar to the second step shows that for any $0\le j\le q$, the images in V^j of all the $v_i^{0,q'}$ ranging over all $q'\ge q$ and relevant i are linearly independent.

Note also that the total number of vectors $v_i^{0,q'}$ with $q'\ge q$ equals

(64)
$$\big(m_{0,q}-m_{0,q+1}\big)+\big(m_{0,q+1}-m_{0,q+2}\big)+\cdots+\big(m_{0,n-1}-m_{0,n}\big)+m_{0,n}=m_{0,q}=\dim\big(V^{0\to q}\big).$$

Hence the part of each $(0,q')$-bar that lives in $V^{0\to q}$ forms a basis there.

Hence we have found a set of $(0,q)$-bars for $q=n,n-1,\dots,0$ whose direct sum equals (59). This is the end of "phase 0."

10.3.4. *Cleaning Up Phase 0.* Note that in phase 0, the $(0,n)$-bars found have a slightly different notation. To clean this up, let $V^{n+1}=0$ and add this into the string $\mathcal{F}=(V^{\cdot},\mathcal{L}^{\cdot})$, yielding

$$V^0\xrightarrow{\mathcal{L}^0}V^1\xrightarrow{\mathcal{L}^1}\cdots\xrightarrow{\mathcal{L}^{n-1}}V^n\xrightarrow{\mathcal{L}^n}V^{n+1}=0. \tag{65}$$

We will call this the *augmented string of vector spaces.* Hence $\mathcal{L}^n$ is the zero map. Then we can extend $\mathcal{L}^{i\to j}$ and $V^{i\to j}$ to $j=n+1$, and $m_{i,n+1}=0$ for all i. Also (62) also makes sense for $q=n$, and as such

$$K_{\ell,n}=\ker(\mathcal{L}^n)\cap\mathcal{L}^{\ell\to n}=\mathcal{L}^{\ell\to n}$$

since $\mathcal{L}^n$ is the zero map. In this way, the basis $w_i^{0,n}$ is equivalently a basis for $K_{0,n}=V^{0\to n}$, and the size of this basis is $m_{0,n}=m_{0,n}-m_{0,n+1}$. This means that the $(0,n)$-bars constructed are constructed from a basis of $K_{0,n}$, just as the $(0,q)$-bars are constructred from a basis of $K_{0,q}$.

Hence, with the above conventions, the $(0,n)$-bar construction is identical to the $(0,q)$-bar construction for $q\le n-1$.

10.3.5. *Phase 1 and Beyond, and the General Step.* At this point we have found $(0,q)$-bars for all $0\le q\le n$ whose direct sum is (59), and this is the end of phase 0.

The next phase, "phase 1," is to add the appropriate $(1,q)$-bars for all $1\le q\le n$; these additional bars, together with the $(0,q)$-bars, should therefore have a direct sum equal to

$$V^0\to V^1\to V^{1\to 2}\to\cdots\to V^{1\to n} \tag{66}$$

At this point we may as well describe "phase ℓ" for any $\ell\ge 1$.

For $\ell=0,1,\dots,n$ the "ℓ-th" phase can be described as follows: consider the two (augmented) sequences:

(67)
$$V^0\to V^1\to\cdots V^{\ell-1}\to V^{\ell-1\to\ell}\to V^{\ell-1\to\ell+1}\to\cdots\to V^{\ell-1\to n}\to 0=V^{\ell-1\to n+1},$$

(68)
$$V^0\to V^1\to\cdots V^{\ell-1}\to V^{\ell}\to V^{\ell\to\ell+1}\to\cdots\to V^{\ell\to n}\to 0=V^{\ell\to n+1}.$$

Before phase ℓ begins, we have found (ℓ', q)-bars for all $\ell' \le \ell - 1$ and all $q \ge \ell'$ whose direct sum equals (67). During phase ℓ we proceed find (ℓ, q)-bars for $q = n, n-1, \ldots, \ell$ — in this order — whose direct sum, along with the bars found before phase ℓ, equals (68).

The following definition is convenient.

Definition 10.14. Let $U' \subset U$ be vector spaces (over an arbitrary field). A *basis of U relative to U'* refers to any set $u_1, \ldots, u_m \in U$, with $m = \dim(U) - \dim(U')$ that satisfy any of the equivalent conditions:

(1) any basis of U' combined with $u_1, \ldots, u_m$ yields a basis for U;
(2) the vectors in U/U' given by $u_1 + U', \ldots, u_m + U'$ are a basis for U/U';
(3) any vector in U can be written uniquely as a vector in U' plus a linear combination of $u_1, \ldots, u_m$.

(It is possible to give a number of other equivalent conditions in the above definition.)

For $j \le \ell - 1$, j-th vector space in (67) and in (68) are both equal to V^j; for $j \ge \ell$, the j-th vector space in (67) and in (68), respective, are

$$V^{\ell-1\to j},\ V^{\ell\to j}$$

respectively. Hence we can rephrase the task for phase ℓ in terms of a relative basis: we want to add (ℓ, q)-bars for each q with $\ell \le q \le n$ so that for each $j \ge \ell$, the sum of the V^j element of the (ℓ, q) bars for all $q \ge j$ is a relative basis for $V^{\ell\to j}$ to $V^{\ell-1\to j}$.

The (ℓ, q) bars for all $q \ge \ell$ that we add can be described as follows: by surjectivity of (65), we have $V^{\ell-1,q} \subset V^{\ell,q}$ since

$$V^{\ell-1\to q} = \mathcal{L}^{\ell-1\to q} V^{\ell-1} = \mathcal{L}^{\ell\to q} \mathcal{L}^{\ell-1} V^{\ell-1} \subset \mathcal{L}^{\ell\to q} V^{\ell} = V^{\ell,q}.$$

It follows that $K_{\ell-1,q} \subset K_{\ell,q}$, since

$$K_{\ell-1,q} = \ker(\mathcal{L}^q) \cap V^{\ell-1,q} \subset \ker(\mathcal{L}^q) \cap V^{\ell,q} = K_{\ell,q}.$$

So choose a basis $\{w_i^{\ell,q}\}_i$ for $K_{\ell,q}$ relative to $K_{\ell-1,q}$, where i ranges from 1 to

$$k_{\ell,q} \stackrel{\text{def}}{=} \dim\bigl(K_{\ell,q}\bigr) - \dim\bigl(K_{\ell-1,q}\bigr) = \bigl(m_{\ell,q} - m_{\ell,q+1}\bigr) - \bigl(m_{\ell-1,q} - m_{\ell-1,q+1}\bigr).$$

We then choose $v_i^{\ell,q} \in V^\ell$ such that $\mathcal{L}^{\ell\to q} v_i^{\ell,q} = w_i^{\ell,q}$.

Hence phase ℓ ends by building all these (ℓ, q)-bars, with q ranging over $\ell \le q \le n$. We now prove that for any $j \ge \ell$, the sum of the V^j parts of all (ℓ, q)-bar vectors over all $q \ge j$ forms a relative basis for $V^{\ell\to j}$ relative to $V^{\ell-1\to j}$.

So fix $j \ge \ell$. We have

$$\sum_{q\ge j} k_{\ell,q} = \sum_{q\ge j} \bigl(m_{\ell,q} - m_{\ell,q+1}\bigr) - \sum_{q\ge j} \bigl(m_{\ell-1,q} - m_{\ell-1,q+1}\bigr)$$

which, similarly to (64), equals

$$m_{\ell,q} - m_{\ell-1,q} = \dim\bigl(V^{\ell\to q}\bigr) - \dim\bigl(V^{\ell-1\to q}\bigr).$$

Hence

$$\sum_{q\ge j} k_{\ell,q} = \dim\bigl(V^{\ell\to q}\bigr) - \dim\bigl(V^{\ell-1\to q}\bigr). \tag{69}$$

Hence the number of (ℓ, q)-bars with $q \ge j$ is the number of vectors in a basis for $V^{\ell,j}$ relative to $V^{\ell-1,j}$. Hence it suffices to show that the vectors $\mathcal{L}^{\ell\to j}(v_i^{\ell,q})$ summed over all $q \ge j$ and appropriate i are linearly independent relative to $V^{\ell-1,j}$.

So say that

$$\sum_{q\ge j} \sum_{i=1}^{k_{\ell,q}} \alpha_i^q \mathcal{L}^{\ell\to j}\big(v_i^{\ell,q}\big) \in V^{\ell-1,j}$$

for some scalars α_i^q. Applying $\mathcal{L}^{j\to n}$ we have

$$\sum_{i=1}^{k_{\ell,n}} \alpha_i^n \mathcal{L}^{\ell\to n}\big(v_i^{\ell,n}\big) \in V^{\ell-1,n}$$

so

$$\sum_{i=1}^{k_{\ell,n}} \alpha_i^n w_i^{\ell,n} \in V^{\ell-1,j}$$

and hence $\alpha_i^n = 0$ for all i, since the $w_i^{\ell,n}$ form a basis of $V^{\ell,j}$ relative to $V^{\ell-1,j}$. We then apply $\mathcal{L}^{j\to n-1}$ and conclude that all $\alpha_i^{n-1} = 0$; similarly we inductively apply $\mathcal{L}^{j\to n'}$ for $n' = n-2, n-3, \ldots, \ell$ to show that $\alpha_i^{n'} = 0$ for all i.

Hence for any $j \ge \ell$, the sum of the V^j parts of all (ℓ, q)-bar vectors over all $q \ge j$ are linearly independent as vectors in $V^{\ell\to j}/V^{\ell-1\to j}$; in view of (69), we have that these V^j parts form a basis for $V^{\ell,j}$ relative to $V^{\ell-1,j}$.

10.3.6. *All Phases, All Steps in Parallel.* Now we observe that for fixed $\ell \le q$, the (ℓ, q)-bars constructed above only need to know $K_{\ell,q}$ and $K_{\ell,q-1}$; in view of (62), $K_{\ell,q}$ depends only on the kernel of $\mathcal{L}_q$ and $V^{\ell,q} = \mathrm{Image}(\mathcal{L}^{\ell\to q})$. Hence from $\mathcal{F} = (V^{\cdot}, \mathcal{L}^{\cdot})$ we can determine $K_{\ell,q}$ and $K_{\ell-1,q}$. In this sense we can find all (ℓ, q)-bars for any ℓ, q using a procedure that is independent of all other choices of the other bars.

10.4. **A Backward Sweep Algorithm.** There is, of course, a corresponding "backwards sweep" algorithm. Perhaps the conceptually simplest way to understand this is that passing to the dual spaces we get the "dual string"

$$(V^n)^* \xrightarrow{(\mathcal{L}^{n-1})^*} (V^{n-1})^* \xrightarrow{(\mathcal{L}^{n-2})^*} \cdots \xrightarrow{(\mathcal{L}^0)^*} (V^0)^*$$

(practically speaking, the matrix representation of $(\mathcal{L}^i)^*$ is the transpose of that of $\mathcal{L}^i$, assuming we've chosen bases for the V^i and we use the corresponding dual basis for each $(V^i)^*$). Then given a bar decomposition for the dual string, we get a basis for each $(V^i)^*$; taking the dual basis for V^i, we easily check that for any $(n-j, n-i)$-bar in the dual string (so $i \le j$)

$$(v^j)^* \to (v^{j-1})^* \to \cdots \to (v^i)^*$$

with $(v^\ell)^* \in (V^\ell)^*$ for $i \le \ell \le j$, then the dual vector $v_\ell \in V^\ell$ for $i \le \ell \le j$ satisfies $\mathcal{L}^\ell v_\ell = v_{\ell+1}$ for all $i \le \ell \le j-1$ (EXERCISE). It follows that a bar decomposition of the dual string gives rise to one of the original string. However, since the order of the bars is reversed, a bar in the dual string

$$(v^j)^* \to (v^{j-1})^* \to \cdots \to (v^i)^*$$

can be seen as starting with $(v^j)^* \in (V^j)^*$ and determining $(v^\ell)^*$ as $(\mathcal{L}^{\ell\to j})^*((v^j)^*)$. In this way the "forward sweeps" in the dual string become "backward sweeps" in the original string.

10.5. **A Barcode Algorithm Based on Jordan Canonical Form.**

Definition 10.15. Consider a string $\mathcal{F} = (V^\cdot, \mathcal{L}^\cdot)$ of vector spaces:

$$V^0 \xrightarrow{\mathcal{L}^0} V^1 \xrightarrow{\mathcal{L}^1} \cdots \xrightarrow{\mathcal{L}^{n-1}} V^n.$$

The *total space of* $\mathcal{F}$ is the pair $(\mathbf{V}, \mathcal{L})$ where

$$\mathbf{V} = V^0 \oplus \cdots \oplus V^n,$$

and $\mathcal{L} \colon \mathbf{V} \to \mathbf{V}$ is the map given by

$$\mathcal{L}(v_0, \ldots, v_n) \to (0, \mathcal{L}^0 v_0, \ldots, \mathcal{L}^{n-1} v_{n-1}).$$

Hence, in the definition above, $\mathcal{L}$ is nilpotent, i.e., $\mathcal{L}^n = 0$.

We next recall the algorithm to put $\mathcal{L}$ into its Jordan canonical form, which determines the barcode of $\mathcal{F} = (V^\cdot, \mathcal{L}^\cdot)$.

10.5.1. *Jordan Canonical Form of a Nilpotent Linear Operator.* In this section we review Jordan canonical form linear operator $\mathcal{L} \colon V \to V$, assuming that $\mathcal{L}$ is nilpotent, i.e., $\mathcal{L}^k = 0$ for some $k \in \mathbb{N}$. We will then apply this to the linear transformation $\mathcal{L} \colon \mathbf{V} \to \mathbf{V}$ in Definition 10.15.

[The general case of Jordan canonical form is not much harder, but we won't need it in this article.]

[This article shows the usefulness of Jordan canonical form in certain "applied settings," despite the fact that "almost all" matrices are diagonalizable.[19]]

So let $\mathcal{L} \colon V \to V$ be a linear transformation of an $\mathbb{R}$-vector space V. Further assume that $\mathcal{L}$ is nilpotent, i.e., $\mathcal{L}^k = 0$ for some $k \in \mathbb{N}$, and fix k to be the smallest such integer. It follows that all eigenvalues of $\mathcal{L}$ are 0; since $0 \in \mathbb{R}$ whether or not $\mathbb{R}$ is algebraically closed, we can put $\mathcal{L}$ into Jordan canonical form (whether or not $\mathbb{R}$ is algebraically closed). Let us review the algorithm.

A *Jordan chain*[20] *of length* k *generated by* w *of* $\mathcal{L}$ is any sequence

$$w, \mathcal{L}w, \ldots, \mathcal{L}^{k-1} w \tag{70}$$

such that $w \in V$ and $\mathcal{L}^{k-1} w \neq 0$. Then it is almost immediate that these elements are linearly independent: indeed, if $\alpha_0 w + \alpha_1 \mathcal{L} w + \cdots + \alpha_{k-1} \mathcal{L}^{k-1} w = 0$ for $\alpha_i \in \mathbb{R}$, and some $\alpha_i \neq 0$, then for the smallest i with $\alpha_i \neq 0$ we apply $\mathcal{L}^{k-1-i}$ to both sides of the equation and conclude that $\alpha_i \mathcal{L}^{k-1} w = 0$, which is impossible.

Next note that for a Jordan chain (70), if we restrict $\mathcal{L}$ to the subspace, V', of V spanned by $\mathcal{L}^{k-1} w, \mathcal{L}^{k-2} w, \ldots \mathcal{L} w, w$, then these vectors, in this order, give a basis for V' that turns $\mathcal{L}|_{V'}$ (the restriction to V') into the matrix $J \in \mathbb{R}^{n \times n}$ acting on

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[19] A matrix $M \in \mathbb{R}^{n \times n}$ (i.e., an $n \times n$ matrix with entries in $\mathbb{R}$), M, is necessarily diagonalizable (over $\overline{\mathbb{R}} = \mathbb{C}$, the algebraic closure of $\mathbb{R}$) when its characteristic polynomial $p_M(x) = \det(Ix - M)$ has n distinct roots, i.e., its discriminant (i.e., the resultant of p_M and p'_M) is nonzero. [The 2×2 all zeros matrix is diagonalizable, and has characteristic polynomial x^2, so this condition is not necessary.] Hence there is a polynomial $Q = Q(M)$, of the entries of M, such that $Q(M) \neq 0$ implies that M is diagonalizable. Since $Q(M)$ is not identically zero (it is nonzero on a diagonal matrix with distinct diagonal elements in $\overline{\mathbb{R}} = \mathbb{C}$), it follows that Q is not the zero polynomial. It follows that the set of non-diagonalizable matrices in $\mathbb{R}^{n \times n}$ is of measure 0. Moreover, the set of non-diagonalizable matrices in $\overline{\mathbb{R}}^{n \times n} = \mathbb{C}^{n \times n}$ lies in a proper, Zariski closed subset, and therefore is "exceptional" in various senses that we will not bother to specify. Similar remarks hold with $\mathbb{R}$ replaced with any field, and $\overline{\mathbb{R}} = \mathbb{C}$ replaced with algebraic closure of the field.

[20] We would add "with respect to the eigenvalue 0" for a general $\mathcal{L}$, not assumed to be nilpotent, and the chain would be $w, (\mathcal{L} - \lambda) w, \ldots, (\mathcal{L} - \lambda)^{k-1} w$ for an eigenvalue λ.

column vectors (i.e., acting to the left of column vectors) where $J = J_k(0) \in \mathbb{R}^{k\times k}$ is the standard $k \times k$ Jordan block matrix for the eigenvalue λ, i.e.,

$$J_k(\lambda) \stackrel{\text{def}}{=} \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix} \tag{71}$$

(where a blank space implies a 0).

It follows that to write $\mathcal{L}$ in Jordan canonical form is to find $w_1, \ldots, w_s \in V$ and $k_1, \ldots, k_s \in \mathbb{N}$ such that (1) for each i, w_i generates a Jordan chain of length k_i, (2) $k_1 + \cdots + k_s = n$, and (3) the union of

$$\bigcup_{i=1}^{s} \{w_i, \mathcal{L}w_i, \ldots, \mathcal{L}^{k_i-1}w_i\}$$

is a basis for V.

Definition 10.16. By a *Jordan basis* for a nilpotent linear operator $\mathcal{L}\colon V \to V$ we mean any pair of sequences $w_1, \ldots, w_s \in V$ and $k_1, \ldots, k_s \in \mathbb{N}$ satisfying (1)–(3) in the previous paragraph.

Example 10.17. Let

$$L = \begin{bmatrix} 0 & 1 & \\ 0 & 0 & \\ & & 0 \end{bmatrix}$$

(where a blank space implies a 0). Then L is a block diagonal matrix with a $J_2(0)$ block and a $J_1(0)$ block. If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the standard basis vectors, then $\mathbf{e}_1, \mathbf{e}_3$ are eigenvectors (with L acting to the left of column vectors). Also, $\mathbf{e}_2, \mathcal{L}\mathbf{e}_2$ and $\mathbf{e}_3$ are two Jordan chains, where $\mathcal{L}$ is the operator on $\mathbb{R}^3$ expressed as column vectors via the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

We say that a Jordan chain in (70) is *maximal* if $\mathcal{L}^k w = 0$ and if there is no w' such that $\mathcal{L}w' = w$. It is easy to see — and helpful for intuition — to note that any Jordan basis for $\mathcal{L}$ must consist of maximal Jordan chains.

It is now easy to give an algorithm for finding a Jordan basis for a nilpotent operator $\mathcal{L}\colon V \to V$. The point is that you want to find the longest Jordan chains of the Jordan basis first.

So let $k \in \mathbb{N}$ be the largest integer with $\mathcal{L}^{k-1}$ nonzero. Let $u_1, \ldots, u_r$ be a basis for the image of $\mathcal{L}^{k-1}$; then, by definition, there exist $w_1, \ldots, w_r$ such that $\mathcal{L}^{k-1}w_i = u_i$ for all i. We easily see that each w_i generates a (maximal) chain of length k, and the vectors $B_k = \{\mathcal{L}^j w_i\}$ with i, j ranging over $1 \le i \le s$ and $0 \le j \le k-1$ are linearly independent, similarly to the above argument. Since $\mathcal{L}^k = 0$, we see that all chains are of length at most k, and that any chain of length k is generated by a w that is a linear combination of $w_1, \ldots, w_r$ above (since $\mathcal{L}^{k-1}w$ must be a linear combination of $u_1, \ldots, u_r$ above).

We next find the maximal chains of length $k-1$ whose elements are linearly independent from B_k above: consider the image of $\mathcal{L}^{k-2}$, which clearly contains $\mathcal{L}^{k-2}w_i$ and $\mathcal{L}^{k-1}w_i$ for all $1 \le i \le r$; since these $2r$ vectors are linearly independent, we can choose $u'_1, \ldots, u'_{r'-1}$ to complete these vectors to a basis for the image of $\mathcal{L}^{k-2}$; we then choose w'_i such that $\mathcal{L}^{k-2} w'_i = u'_i$ for all i. We easily show that

$$B_k = \{\mathcal{L}^j w_i \mid 1 \le i \le r,\ 0 \le j \le k-1\}$$

and

$$B_{k-1} = \{\mathcal{L}^j w'_i \mid 1 \le i \le r',\ 0 \le j \le k-2\}$$

are disjoint subsets whose union is linearly independent.

Next we repeat this step to find vectors B_{k-2}, independent of $B_k \cup B_{k-1}$ and coming from chains of length $k-2$. We similarly find vectors $B_{k-3}, B_{k-4}, \ldots B_1$. Since B_1 is a basis of vectors in the image of $\mathcal{L}^0 = \mathrm{id}_V$, i.e., all of V, that completes $B_2 \cup \ldots \cup B_k$, we have that $B_1 \cup \ldots \cup B_k$ is a basis for all of V. The union over all i of the chains of length i arising from the B_i is therefore a Jordan basis.

Example 10.18. In Example 10.17, the image of $\mathcal{L}$ on $\mathbb{R}^3$ is the span of $\mathbf{e}_1$; since $\mathcal{L}\mathbf{e}_2 = \mathbf{e}_1$, this gives us the chain $\mathbf{e}_2, \mathcal{L}\mathbf{e}_2 = \mathbf{e}_1$, so $B_2 = \{\mathbf{e}_2, \mathbf{e}_1\}$. Moreover, B_2 is determined up to scalar multiple. Then B_1 consists of a single element, which may be any vector $\gamma_1\mathbf{e}_1 + \gamma_3\mathbf{e}_3$ with $\gamma_1, \gamma_3 \in \mathbb{R}$ with $\gamma_3 \neq 0$. Notice that if we started by looking for Jordan chains of length 1, i.e., eigenvectors, there is 2-dimensional possible space. If we take $\mathbf{e}_1 + \mathbf{e}_3$ and $\mathbf{e}_3$ as such a basis, there is no way to extend either of these "backwards" to make one of them a chain of length 2. This is why we start by finding the longest Jordan chains and then find successively shorter ones.

Remark 10.19. Despite the problem arising by starting with shorter chains and extending them backwards, identified in Example 10.18, one can still roughly do this, using one trick. Namely, in Section 3.1 of [HJ85] one first finds a basis with respect to which $\mathcal{L}$ is an upper triangular matrix, using the Schur decomposition; hence the diagonal is all 0's. From there one does an inductive argument, reducing the $n \times n$ case (assuming the matrix is upper triangular with 0's on the diagonal) to the $(n-1)\times(n-1)$ case (see Subsection 3.1.5 there). So provided that $\mathcal{L}$ is already written in upper triangular form, one can start with short chains and progressively look for longer (or new) ones.

Remark 10.20. Note that the total space $(\mathbf{V}, \mathcal{L})$ of a string of vector spaces is already an upper triangular matrix: indeed, choose arbitrary bases $B^0, \ldots, B^n$ for the respective vector spaces $V^0, \ldots, V^n$; then $\mathcal{L}$ with respect to $B^n, \ldots, B^0$ is a block matrix whose only nonzero blocks are those just above (or to the right of) the main diagonal. Hence in Remark 10.19 we can skip the Schur decomposition step.

Remark 10.21. There are likely very many algorithms to find a barcode decomposition of a string of vector spaces, and I currently (December 2024) don't know what is known here for a general string and/or strings arising in homology. However, given the previous two remarks, I'm guessing there are a lot of options, depending on the precise features of the string of vector spaces.

10.5.2. *Proof of the Barcode Theorem.* Let notation be as in Definition 10.15. Since $\mathcal{L}$ is nilpotent, we will use the algorithm in the previous section. So let $k \in \mathbb{N}$ be the smallest natural number with $\mathcal{L}^k = 0$, and let $\mathbf{u}_1, \ldots, \mathbf{u}_r$ be a basis for the image of $\mathcal{L}^{k-1}$; let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be such that $\mathbf{u}_i = \mathcal{L}^{k-1}\mathbf{w}_i$.

Say that a nonzero element $\mathbf{u} \in \mathbf{V}$ is *purely of degree d* if $\mathbf{u} = (u^0, \ldots, u^n)$ and $u^i = 0$ for $i \neq d$. Clearly any nonzero element of $\mathbf{V}$ can be uniquely written as a sum of elements purely of degrees $d_1 < d_2 < \ldots < d_t$ with $0 \le d_1 < \ldots < d_t \le n$; we call each such summand a *pure component of* $\mathbf{u}$. Clearly if $\mathbf{u}_i$ is in the image of $\mathcal{L}^{k-1}$, then so is each pure component of $\mathbf{u}_i$. It follows from the "basis exchange theorem" that we can replace the basis $\mathbf{u}_1, \ldots, \mathbf{u}_r$ of the image of $\mathcal{L}^{k-1}$ with one where each $\mathbf{u}_j$ is purely of some degree. Then if $\mathcal{L}^{k-1}\mathbf{w}_j = \mathbf{u}_j$ and $\mathbf{u}_j$ is purely of degree d, then the same holds with $\mathbf{w}_j$ replaced by its pure component of degree $d-(k-1)$.

This gives us a set B_k as in the previous section, which is the union of chains of length k each of which is generated by a w_i that is purely ofsome degree. We easily see that for any d, the dimension of the image of V^d in V^{d+k-1} is precisely the number of w_i that are purely of degree d; hence the number of $\mathbf{w}_1, \ldots, \mathbf{w}_r$ that are purely of some degree d depends only on d and not the particular choice of $\mathbf{w}_1, \ldots, \mathbf{w}_r$.

In this way we similarly generate $B_{k-1}, B_{k-2}, \ldots, B_1$, which give a Jordan basis for $\mathcal{L}$. Moreover the number of Jordan chains of a given length k' in $B_{k'}$ beginning in an element purely in any given degree is independent of the choice of elements in $B_{k'}$. But the decomposition of $\mathbf{V}$ into Jordan chains generated by elements of $\mathbf{V}$, each of which is purely of some degree, is clearly the same thing as a barcode decomposition.

Part 4. Exercises and Appendices

Appendix A. Exercises for January 2025 and February 2025

A.1. Exercises for January 2025.

Some of the EXERCISES in the main text will actually be done in class. Here we gather some exercises to be handed in, which includes some additional exercises.

Exercise A.1. Show that if $\{\mathbf{a}_0, \ldots, \mathbf{a}_d\}$ are in general position, then for any $\mathbf{b}$ satisfying

$$\mathbf{b} = \alpha_0\mathbf{a}_0 + \cdots + \alpha_d\mathbf{a}_d,$$

for any $\alpha_i \in \mathbb{R}$ such that $\alpha_0 + \cdots + \alpha_d = 1$, then these α_i (satisfying $\alpha_0 + \cdots + \alpha_d = 1$) are unique.[21]

Exercise A.2. Prove that if $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^N$ are any three vectors in generalized position, and if

$$\mathrm{Conv}(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2) = \mathrm{Conv}(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2),$$

then

$$\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2\} = \{\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2\}.$$

Do this by proving that none of $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2$ equals $\mathbf{a}_0$, then $\mathbf{a}_0$ cannot lie in $\mathrm{Conv}(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2)$. [Hint: It may help to express $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2$ in barycentric coordinates.] Then explain how to modify this argument to prove the analogous result where $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ is replaced by any arbitrary set $\mathbf{a}_0, \ldots, \mathbf{a}_d \in \mathbb{R}^N$ in general position.

[21] For simplicies, we are interested in non-negative α_i; however, for this exercise we don't need to assume that the α_i are non-negative.

Exercise A.3. For $i \in \mathbb{N} = \{1, 2, \ldots\}$, let $\mathbf{x}_i = (i, i^2, i^3) \in \mathbb{R}^3$. Show that for any distinct $i, j, k, \ell \in \mathbb{N}$ we have that $\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k, \mathbf{x}_\ell$ are in general position. You may use the fact that any Vandermonde matrix, such as a 4×4 matrix of the form

$$\begin{bmatrix} 1 & a_1 & a_1^2 & a_1^3 \\ 1 & a_2 & a_2^2 & a_2^3 \\ 1 & a_3 & a_3^2 & a_3^3 \\ 1 & a_4 & a_4^2 & a_4^3 \end{bmatrix}$$

with $a_1, \ldots, a_4$ distinct is invertible, i.e., has nonzero determinant.

Exercise A.4. Give an equivalent condition for vectors $\mathbf{x}_0, \ldots, \mathbf{x}_N$ in $\mathbb{R}^N$ to be in general position, in terms of an $(N+1) \times (N+1)$ matrix (and prove that your condition is equivalent). (Exercise A.3 indicates the condition.)

Exercise A.5. Let $\mathsf{K}_{\mathsf{abs}}$ be an abstract simplicial complex of dimension at most D, i.e., each set in $\mathsf{K}_{\mathsf{abs}}$ has at most $D+1$ elements. Show that there is a simplicial complex $S \subset \mathbb{R}^{2D+1}$ whose associated abstract simplicial complex is $\mathsf{K}_{\mathsf{abs}}$. [Hint: In class we did the case $D = 1$. First show that if $X = \{\mathbf{x}_0, \ldots, \mathbf{x}_r\}$ and $X' = \{\mathbf{x}'_0, \ldots, \mathbf{x}'_s\}$ are sets in $\mathbb{R}^{2D+1}$ such that $X \cup X'$ are in general position, then

$$\operatorname{conv}(X) \cap \operatorname{conv}(X') = \operatorname{conv}(X \cap X').$$

[Notice that it is very easy to see that $\operatorname{conv}(X) \cap \operatorname{conv}(X')$ contains $\operatorname{conv}(X \cap X')$; what you really have to show is that if $\operatorname{conv}(X) \cap \operatorname{conv}(X')$ can't contain anything more than $\operatorname{conv}(X \cap X')$, assuming that $X \cup X'$ are in general position.] In class we did this by hand in all the special cases. To do this more systematically, you could consider the result in Exercise A.1.]

Exercise A.6. In this exercise we compute the homology groups of $\mathsf{K}^0_{\mathsf{abs}}, \mathsf{K}^1_{\mathsf{abs}}, \mathsf{K}^2_{\mathsf{abs}}$ in Example 4.9. For calculations regarding ∂_1, analogous computations were performed for a complete graph on four vertices (hence ∂_1 is a bit different) in Example 4.4; this should serve as a model for the first parts of this exercise.

A.6(a) Write down the matrix M that represents ∂_1 of $\mathsf{K}^0_{\mathsf{abs}}$ in Example 4.9, with respect to the basis for $\mathcal{C}_0(\mathsf{K}^0_{\mathsf{abs}})$

$$[A], [B], [C], [D] \tag{72}$$

and basis for $\mathcal{C}_1(\mathsf{K}_{\mathsf{abs}})$

$$[A, B], [A, C], [A, D], [B, C], [C, D]. \tag{73}$$

A.6(b) Determine the row reduced echelon form of the matrix M for ∂_1.

A.6(c) Explicitly determine $Z_1 = \ker(\partial_1)$ for $\mathsf{K}^0_{\mathsf{abs}}$, as a formula

$$f_1(\alpha_1, \alpha_2)[A, B] + \cdots + f_5(\alpha_1, \alpha_2)[C, D]$$

where $f_i(\alpha_1, \alpha_2)$ are linear functions.

A.6(d) Determine $B_0 = \operatorname{Image}(\partial_1)$, by computing the column reduced echelon form of M, and then using it write a formual

$$g_1(\boldsymbol{\beta})[A] + g_2(\boldsymbol{\beta})[B] + g_3(\boldsymbol{\beta})[C] + g_4(\boldsymbol{\beta})[D]$$

where $g_1, \ldots, g_4$ are linear functions of a vector $\boldsymbol{\beta}$ whose number of parameters is the rank of M. [Hint: See Example 4.4.]

A.6(e) Determine β_0, β_1 of $\mathsf{K}^0_{\mathsf{abs}}$.

A.6(f) Write down the matrix that represents ∂_2 of $\mathsf{K}^1_{\mathsf{abs}}$ with respect to the basis $[A, B, C]$ of $\mathcal{C}_2(\mathsf{K}^1_{\mathsf{abs}})$ and the above basis (73) for $\mathcal{C}_1(\mathsf{K}^1_{\mathsf{abs}})$.

A.6(g) Show that $\ker(\partial_2(\mathsf{K}^1_{\mathsf{abs}})) = 0$. (Thanks to a student who came to office hours on Feb 6, 2025; the original (incorrect) version read "$\ker(\partial_2(\mathsf{K}^1_{\mathsf{abs}})) = \mathcal{C}_2(\mathsf{K}^1_{\mathsf{abs}})$.") What is $\beta_2(\mathsf{K}^1_{\mathsf{abs}})$?

A.6(h) Show that for any two-dimensional complex, $\mathsf{K}_{\mathsf{abs}}$, we have

$$\beta_0 - \beta_1 + \beta_2 = |V| - |E| + |F|,$$

where F is the set of 2-faces of $\mathsf{K}_{\mathsf{abs}}$. [Hint: See Example 4.4.] The left-hand-side above is called the *Euler characteristic of K_{abs}*.

A.6(i) Note that $\partial_1(\mathsf{K}^i_{\mathsf{abs}})$ depends only on the underlying graph of $\mathsf{K}^i_{\mathsf{abs}}$, and hence they are all the same. What is $\beta_0(\mathsf{K}^i_{\mathsf{abs}})$?

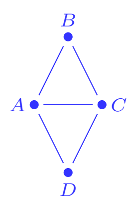
A.6(j) Use the previous three parts to determine $\beta_1(\mathsf{K}^1_{\mathsf{abs}})$.

A.6(k) Similarly, determine $\beta_j(\mathsf{K}^2_{\mathsf{abs}})$ for $j = 0, 1, 2$.

Exercise A.7. Let $G = (V, E)$ be a graph. A walk $w = (v_0, \ldots, v_k)$ in G is *closed* if $v_k = v_0$; moreover we say that w is *cyclic* if it is closed and $v_0, \ldots, v_{k-1}$ are distinct vertices (hence w "traverses a cycle in G once"[22]). To any walk (closed, cyclic, or not) $w = (v_0, \ldots, v_k)$, the 1-*form of w* is the element of $\mathcal{C}_1(G)$

$$w_{1-\text{form}} = (v_0, \ldots, v_k)_{1-\text{form}} \stackrel{\text{def}}{=} [v_0, v_1] + \cdots + [v_{k-1}, v_k].$$

For example, consider the graph G given by:



(74) G

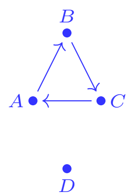
As examples we have

(1) $(A, B, C)_{1-\text{form}} = [A, B] + [B, C] \in \mathcal{C}_1(G)$;
(2) $(A, B, A)_{1-\text{form}} = [A, B] + [B, A] \in \mathcal{C}_1(G)$, which equals 0 in $\mathcal{C}_1(G)$ (since in $\mathcal{C}_1(G)$ we identify $[A, B]$ with $-[B, A]$);
(3) $(A, B, C, A)_{1-\text{form}} = [A, B] + [B, C] + [C, A] \in \mathcal{C}_1(G)$ (which is non-zero in $\mathcal{C}_1(G)$).
(4) $(A, B, C, D, A)_{1-\text{form}} = [A, B] + [B, C] + [C, D] + [D, A] \in \mathcal{C}_1(G)$

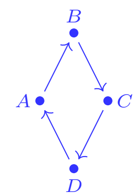
The closed walk in (2) has associated 1-form equal to 0 in $\mathcal{C}_1(G)$. But the closed walks in (3) and (4) has associated 1-form that is non-zero in $\mathcal{C}_1(G)$, and that we

[22]In class we explained that in graph theory, a *cycle (of length k)* refers to a graph with vertices $v_1, \ldots, v_k$, and edges $\{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}, \{v_k, v_1\}$ (when working with simple graphs we must insist that $k \geq 3$), and that a *cycle in G* refers to any subgraph that is a cycle. Note to any walk in a simple graph, $w = (v_0, v_1, \ldots, v_k)$ there is a *reverse walk of w*, $w^{\text{rev}} = (v_k, v_{k-1}, \ldots, v_0)$, and $w^{\text{rev}}_{1-\text{form}} = -w_{1-\text{form}}$. In particular, a cycle graph of length k can be "traversed" by a closed walk of length k in either "direction" and beginning at any vertex; the associated 1-form is unchanged, up to a $\pm$ sign.

depict as:

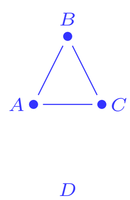


Depiction of $[A, B] + [B, C] + [C, A]$

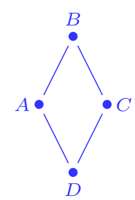


Depiction of $[A, B] + [B, C] + [C, D] + [C, D]$

Notice that these two 1-forms each traverse a subgraph of G that is a *cycle graph*, namely:

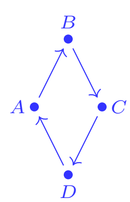


A cycle graph of length 3

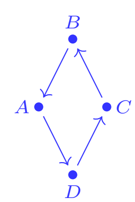


A cycle graph of length 4

Notice also that the "reverse walk" would traverse a cycle in the reverse direction: e.g., the reverse walk of (A, B, C, D, A) is (A, D, C, B, A) (i.e., you reverse the order of the letters), and the reverse walk is depicted by arrows in the reverse orientation:



$(A, B, C, D, A)_{1-\mathrm{form}}$
$= [A, B] + [B, C] + [C, D] + [C, D]$



$(A, D, C, B, A)_{1-\mathrm{form}}$
$= [A, D] + [D, C] + [C, B] + [B, A]$

[Some of the exercises below are really exercises in graph theory.]

A.7(a) Show that if w is a closed walk in G, then its associated 1-form equals the sum of (zero or more) 1-forms associated to cyclic walks in G.

Example: let G be the graph in (74). Here are some examples:

(a) $(A, B, A)_{1-\mathrm{form}} = [A, B] + [B, A] = 0$ in $\mathcal{C}_1(G)$, which equals the sum of zero cyclic walks.

(b) $(A, B, A, B, C, A)_{1-\mathrm{form}} = [A, B] + [B, A] + [A, B] + [B, C] + [C, A] = [A, B] + [B, C] + [C, A]$ in $\mathcal{C}_1(G)$, which equals $(A, B, C, A)_{1-\mathrm{form}}$, and (A, B, C, A) is the cyclic walk in (3) just below (74). Hence

$$(A, B, A, B, C, A)_{1-\mathrm{form}} = (A, B, C, A)_{1-\mathrm{form}}$$

expresses the left-hand-side as a sum of one 1-form associated to a cyclic walk.

(c) Similarly

$$(A,B,C,A,D,C,A)_{1-\mathrm{form}} = (A,B,C,A)_{1-\mathrm{form}} + (A,D,C,A)_{1-\mathrm{form}}$$
$$(A,B,C,D,A,C,B,A)_{1-\mathrm{form}} = (C,D,A,C)_{1-\mathrm{form}}$$

express the left-hand-sides as a sum of 1-forms associated to cyclic walks.

A.7(b) Let $\tau = \sum_{v\neq v'} \alpha_{v,v'}[v,v'] \in \mathcal{C}_1(G)$ satisfy (1) $\alpha_{v,v'} \in \mathbb{Z}_{\geq 0}$ for all $v \neq v'$ (i.e., the $\alpha_{v,v'}$ are non-negative), and (2) $\partial_1\tau = 0$, i.e., $\tau \in Z_1(G)$. Show that there is a set of closed walks $w^1,\ldots,w^r$ such that

$$\tau = w^1_{1-\mathrm{form}} + \cdots + w^r_{1-\mathrm{form}},$$

and for any $v, v' \in V$, the number of times v, v' appears consecutively (in this order) in all these walks is $\alpha_{v,v'}$. [Hint: You can use a "greedy algorithm" plus induction. To get the idea, you could play around with some small examples, e.g., the graph in (74).]

A.7(c) Show that the closed walks $w^1,\ldots,w^r$ in the previous part can be taken to be cyclic walks, provided that for all $v \neq v'$ we have that $\alpha_{v,v'} = 0$ or $\alpha_{v',v} = 0$. [Hint: Again, a "greedy algorithm" will work.]

A.7(d) Show that the previous part is not true if we are allowed to have both $\alpha_{v,v'}, \alpha_{v',v}$ to be nonzero for any $v \neq v'$. [Hint: this is true for any graph with at least one edge.]

A.7(e) Let $\tau = \sum_{v\neq v'} \alpha_{v,v'}[v,v'] \in \mathcal{C}_1(G)$ satisfy (1) $\alpha_{v,v'} \in \mathbb{Z}$ for all $v \neq v'$, and (2) $\partial_1\tau = 0$, i.e., $\tau \in Z_1(G)$. Show that there is a set of cyclic walks $w^1,\ldots,w^r$ such that

$$\tau = w^1_{1-\mathrm{form}} + \cdots + w^r_{1-\mathrm{form}}.$$

A.7(f) Illustrate each of the previous parts with examples that come from $\mathsf{K}^0_{\mathsf{abs}}$ of Example 4.9. (It is your choice to give an example that is not too complicated, but complicated enough to show how the previous parts works.)

A.7(g)* Show that if

$$\tau = \sum_{v\neq v'} \alpha_{v,v'}[v,v'] \in Z_1(G) = \ker(\partial_1) \subset \mathcal{C}_1(G),$$

with $\alpha_{v,v'} \in \mathbb{R}$, there are real $\beta_1,\ldots,\beta_r$ and cyclic walks $w^1,\ldots,w^r$ in G such that

$$\tau = \beta_1 w^1_{1-\mathrm{form}} + \cdots + \beta_r w^r_{1-\mathrm{form}}.$$

[Hint: consider the $\mathbb{Q}$-linear span of the $\alpha_{v,v'}$ in $\mathbb{R}$; this is a $\mathbb{Q}$-vector space in $\mathbb{R}$.]

The above exercise provides the strong intuition that an element of $Z_1(G)$ equals a linear combination of elements of $Z_1(G)$ that arises from cyclic walks in G.

Exercise A.8. Let

$$\mathsf{K}_{\mathsf{abs}} = \mathrm{Power}(\{A,B,C\}) \setminus \bigl\{\{A,B,C\}\bigr\} = \bigl\langle \{A,B\},\{A,C\},\{B,C\}\bigr\rangle$$
$$= \bigl\{\emptyset,\{A\},\{B\},\{C\},\{A,B\},\{A,C\},\{B,C\}\bigr\}$$

(hence $\mathsf{K}_{\mathsf{abs}}$ is a 1-dimensional abstract simplicial complex consisting every proper face (i.e., every proper subset) of the 2-dimensional set $\{A,B,C\}$).

A.8(a) Find a non-zero element $\tau \in Z_1(\mathsf{K}_{\mathsf{abs}}) = \ker(\partial_1)$.

A.8(b) Is this element τ also non-zero in $H_1(\mathsf{K}_{\mathsf{abs}})$? [Recall the notation: $H_1 = Z_1/B_1$, where $B_1 = \mathrm{Image}(\partial_2)$ and Z_1 as above.] Explain.

A.8(c) If $\mathsf{L}_{\mathsf{abs}}$ is $\mathsf{K}_{\mathsf{abs}}$ and the set $\{A, B, C\}$, i.e., $\mathsf{L}_{\mathsf{abs}} = \mathrm{Power}(\{A, B, C\})$, is the τ from the previous part still in $Z_1(\mathsf{L}_{\mathsf{abs}})$? Would this τ still be non-zero in $H_1(\mathsf{L}_{\mathsf{abs}})$? Explain.

A.8(d) More generally, let $i \geq 2$ be an integer, $S = \{A_0, A_1, \ldots, A_i\}$ and $\mathsf{K}_{\mathsf{abs}} = \mathrm{Power}(S) \setminus \{S\}$ ($\mathsf{K}_{\mathsf{abs}}$ is often called an "i-simples without its interior" or "the boundary of an i-simplex"); find a non-zero element τ of $Z_{i-1}(\mathsf{K}_{\mathsf{abs}})$.

A.8(e) Is τ nonzero in $H_{i-1}(\mathsf{K}_{\mathsf{abs}})$? Explain. (We will later prove that for this complex we have $H_j(\mathsf{K}_{\mathsf{abs}}) = 0$ if $j \neq 0, i-1$, and $H_0(\mathsf{K}_{\mathsf{abs}}), H_{i-1}(\mathsf{K}_{\mathsf{abs}})$ are one-dimensional.)

A.8(f) Let $\mathsf{L}_{\mathsf{abs}}$ be $\mathsf{K}_{\mathsf{abs}}$ and the set S, i.e., $\mathsf{L}_{\mathsf{abs}} = \mathrm{Power}(S)$. Is the τ from the previous part still in $Z_{i-1}(\mathsf{L}_{\mathsf{abs}})$? Would this τ still be non-zero in $H_{i-1}(\mathsf{L}_{\mathsf{abs}})$? Explain.

Exercise A.9. Let $\mathsf{K}_{\mathsf{abs}} = \mathrm{Power}(\{A, B, C\}) \setminus \big\{\{A, B, C\}\big\}$ as in the previous exercise. Let $\tau = [A, B] + [B, C] + [C, A]$.

A.9(a) Show that $\partial_1 \tau = 0$.

A.9(b) Let P be distinct from A, B, C. Let $\mathsf{L}_{\mathsf{abs}} = \mathrm{Cone}_P(\mathsf{K}_{\mathsf{abs}})$. Use the first proof of Theorem 6.1 (as done in class on Jan 24, 2025) to find a $\sigma \in \mathcal{C}_2(\mathsf{L}_{\mathsf{abs}})$ such that $\partial_2 \sigma = \tau$.

A.9(c) Let P' be distinct from A, B, C, P, and let

$$\mathbf{S}\mathsf{K}_{\mathsf{abs}} = \mathrm{Cone}_P(\mathsf{K}_{\mathsf{abs}}) \cup \mathrm{Cone}_{P'}(\mathsf{K}_{\mathsf{abs}})$$

(which is called a (or the) *suspension of $\mathsf{K}_{\mathsf{abs}}$*). Using the construction in the previous part, give an element $\tilde{\sigma} \in \mathcal{C}_2(\mathbf{S}\mathsf{K}_{\mathsf{abs}})$ that is non-zero and has $\tilde{\sigma} \in \ker(\partial_2(\mathbf{S}\mathsf{K}_{\mathsf{abs}}))$, i.e., $\partial_2(\tilde{\sigma}) = 0$.

A.9(d) Is $\tilde{\sigma}$ in the previous part equal to 0 in $H_2(\mathbf{S}\mathsf{K}_{\mathsf{abs}})$? Explain.

A.2. Exercises for Februrary 2025.

Exercise A.10. Consider Example 5.2. Let α be the 1-form in $\mathcal{C}_1(G) = \mathcal{C}_1(G_1 \cup G_2)$ given by

$$\beta = (_{1-\mathrm{form}}(A, A'', B, A'', A)) = [A, A''] + [A'', B] + [B, A'] + [A', A]$$

A.10(a) Use the Mayer-Vietoris theorem to compute $\delta(\beta)$.

A.10(b) In class on Feb 12, 2025, we computed $\delta(\beta')$ for

$$\beta' = [A, A'] + [A', B] + [B, A''] + [A'', A],$$

and where G a slightly smaller graph. How does your computation of $\delta(\beta)$ compare with our computation of $\delta(\beta')$ in class?

Exercise A.11. Let K and L be abstract simplicial complexes whose vertex sets are disjoint. The *join of K and L* refers to the simplicial complex

$$\mathsf{K} * \mathsf{L} = \{S_1 \cup S_2 \mid S_1 \in \mathsf{K},\ S_2 \in \mathsf{L}\}.$$

A.11(a) Show that if $\mathsf{L} = \langle\{P\}\rangle$ (i.e., $\mathsf{L} = \{\emptyset, \{P\}\}$ with P therefore disjoint from the vertex set of K), then $\mathsf{K} * \mathsf{L}$ equals $\mathrm{Cone}_P(\mathsf{K})$.

A.11(b) Show that if $\mathsf{L} = \langle\{P_1\}, \{P_2\}\rangle$, then $\mathsf{K} * \mathsf{L}$ is a suspension of K.

A.11(c) Show that if $\mathsf{L} = \langle\{P_1\}, \ldots, \{P_m\}\rangle$ with $P_1, \ldots, P_m$ distinct (and distinct from the vertices of K), then for all $i \in \mathbb{Z}$,

$$\beta_i(\mathsf{K} * \mathsf{L}) = \begin{cases} 1 & \text{if } i = 0, \\ (m-1)\bigl(\beta_0(\mathsf{K}) - 1\bigr) & \text{if } i = 1, \\ (m-1)\beta_{i-1}(\mathsf{K}) & \text{if } i \geq 2, \end{cases}$$

where $\beta_i(\cdot)$ denotes the i-th Betti number (i.e., $\beta_i(\cdot) = \dim(H_i(\cdot))$). [Hint: Use induction on m; the cases $m = 1, 2$ were (or will be) covered in class; use the Mayer-Vietoris sequence.]

Exercise A.12. An abstract simplicial complex K is

(1) an *i-simplex* if it is of the form

$$\mathsf{K} = \mathrm{Power}(S)$$

where S is a set of size $|S| = i + 1$, and

(2) an *i-simplex without its interior* or *the boundary of an i-simplex* if it is of the form

$$\mathsf{K} = \mathrm{Power}(S) \setminus \{S\}$$

where $|S| = i + 1$.

(This is the same terminology as in Exercise A.8). Let $S = \{A_0, \ldots, A_i\}$ be a set of size $i + 1$, and let

$$\mathsf{K} = \mathrm{Power}(S) \setminus \{S\}.$$

Let

$$\mathsf{K}_1 = \mathsf{K} \setminus \{\{A_1, \ldots, A_i\}\},$$

i.e., K_1 is K minus one of its i-faces; let

$$\mathsf{K}_2 = \mathrm{Power}(\{A_1, \ldots, A_i\}).$$

Hence $\mathsf{K}_1 \cup \mathsf{K}_2 = \mathsf{K}$.

A.12(a) Show that $\mathsf{K}_1, \mathsf{K}_2$ are both cones.

A.12(b) Using the Mayer-Vietoris sequence and induction on i, show that for $i \geq 2$, the Betti numbers of K are: $\beta_j(\mathsf{K}) = 1$ for $j = 0, i-1$ (this originally was written: for $j = 0, 1$, and was corrected by QW during office hours, March 6), and otherwise $\beta_j(\mathsf{K}) = 0$. [For $i = 1$ we have $\beta_0(\mathsf{K}) = 2$ and $\beta_i(\mathsf{K}) = 0$ for $i \geq 1$.]

Exercise A.13. Let K_1 be any simplicial complex, and let $\mathsf{K}_2 = \mathrm{Power}(S)$ be an i-simplex (i.e., $|S| = i + 1$) for some $i \geq 1$ such that

(1) $S \notin \mathsf{K}_1$, and
(2) if $S' \subset S$ is any proper subset of S (i.e., $S' \neq S$), then $S' \in \mathsf{K}_1$.

Let $\mathsf{K} = \mathsf{K}_1 \cup \mathsf{K}_2$. (Hence K_1 equals K minus a single i-simplex whose boundary lies entirely in K_1.)

A.13(a) Show that either

$$\beta_{i-1}(\mathsf{K}) = \beta_{i-1}(\mathsf{K}_1) - 1, \quad \text{and} \quad \forall j \neq i-1, \beta_j(\mathsf{K}) = \beta_j(\mathsf{K}_1); \tag{75}$$

or

$$\beta_i(\mathsf{K}) = \beta_i(\mathsf{K}_1) + 1, \quad \text{and} \quad \forall j \neq i, \beta_j(\mathsf{K}) = \beta_j(\mathsf{K}_1). \tag{76}$$

(In other words, in passing from K_1 to K, all Betti numbers are the same except that either (1) β_i increases by one, or (2) β_{i-1} decreases by one.) [Hint: you can (and will need to...) use the result in Exercise A.12; hint added during office hours on March 6, thanks to QY.]

A.13(b) Show that we have

$$\beta_i(\mathsf{K}) - \beta_{i-1}(\mathsf{K}) = \beta_i(\mathsf{K}_1) - \beta_{i-1}(\mathsf{K}_1) + 1. \tag{77}$$

Exercise A.14. Let L be any simplicial complex, and let C_i be the number of i-faces of L. Show that

$$\beta_0(\mathsf{L}) + \beta_1(\mathsf{L}) + \cdots \leq C_0 + C_1 + \ldots.$$

[The left-hand-side is sometimes called the *homological complexity of* L.] [Hint: use a previous exercise.]

Do either Exercise A.15 or Exercise A.16.

Exercise A.15. Let L be any simplicial complex, and let C_i be the number of i-faces of L. Use (77) to show the following.

A.15(a) $\beta_0(\mathsf{L}) \leq C_0$.

A.15(b) $\beta_1(\mathsf{L}) - \beta_0(\mathsf{L}) \leq C_1 - C_0$. [Hint: let L^0 consist of only the 0-faces (or vertices) of L, and let $\mathsf{L}^{\leq 1}$ consist of the 0-faces and 1-faces of L; we have

$$C_1 = \bigl(\beta_1(\mathsf{L}^{\leq 1}) - \beta_0(\mathsf{L}^{\leq 1})\bigr) - \bigl(\beta_1(\mathsf{L}^0) - \beta_0(\mathsf{L}^0)\bigr).$$

What do we know about $\beta_1(\mathsf{L}^0)$ and $\beta_0(\mathsf{L}^0)$? What can you say about

$$\beta_1(\mathsf{L}) - \beta_0(\mathsf{L}) \quad \text{versus} \quad \beta_1(\mathsf{L}^{\leq 1}) - \beta_0(\mathsf{L}^{\leq 1}) \ ?$$

]

A.15(c) More generally, show that for any d:

$$\beta_d(\mathsf{L}) - \beta_{d-1}(\mathsf{L}) + \cdots + (-1)^d \beta_0(\mathsf{L}) \leq C_d - C_{d-1} + \ldots + (-1)^d C_0.$$

Exercise A.16. Let L be any simplicial complex, and let C_i be the number of i-faces of L. For an integer $i \geq 0$, let $\mathsf{L}^{\leq i}$ denote the union of all j-faces in L with $j \leq i$. Hence $\mathsf{L}^{\leq 0}$ is the set of vertices of L (which we also write as L^0), and for each $i \geq 1$, $\mathsf{L}^{\leq i}$ is obtained from $\mathsf{L}^{\leq i-1}$ by adding C_i i-faces. Say that we add these C_i i-faces to $\mathsf{L}^{\leq i-1}$ in some order, and let C_i' be the number of i-faces such that (76) holds, and C_i'' be the number of i-faces such that (75) holds. Hence $C_i = C_i' + C_i''$.

A.16(a) Show that

$$\beta_i(\mathsf{L}^{\leq i}) = C_i',$$

and

$$\beta_{i-1}(\mathsf{L}^{\leq i}) = \beta_{i-1}(\mathsf{L}^{\leq i-1}) - C_i'',$$

and

$$\forall j \neq i, i-1, \quad \beta_j(\mathsf{L}^{\leq i}) = \beta_j(\mathsf{L}^{\leq i-1})$$

[It follows that C'_i, C''_i do not depend on the order in which we attach the i-faces.]

A.16(b) Conclude that for all i,

$$\beta_i(\mathsf{L}) = C'_i - C''_{i+1}.$$

A.16(c) Use these results to give another proof that for any integer $d \geq 0$,

$$\beta_d(\mathsf{L}) - \beta_{d-1}(\mathsf{L}) + \cdots + (-1)^d \beta_0(\mathsf{L}) \leq C_d - C_{d-1} + \ldots + (-1)^d C_0.$$

Exercise A.17. Recall that if $\mathcal{L}\colon U \to W$ is a linear map of finite dimensional vector spaces, then we define

$$\operatorname{rank}(\mathcal{L}) \stackrel{\text{def}}{=} \dim\bigl(\operatorname{Image}(\mathcal{L})\bigr),$$

and we have

$$\dim(U) = \operatorname{rank}(\mathcal{L}) + \dim\bigl(\ker(\mathcal{L})\bigr).$$

Let

$$0 \xrightarrow{d_{m+1}} V_m \xrightarrow{d_m} \cdots \xrightarrow{d_1} V_0 \xrightarrow{d_0} 0$$

be a finite chain, (implying that $V_i = 0$ for $i \geq m+1$ and $i \leq -1$). Assume that each V_i is finite dimensional, and let β_i be the i-th Betti number of $(\mathbf{V}, \mathbf{d})$ (see (22)).

A.17(a) Show that

$$\beta_0 - \beta_1 + \ldots + (-1)^m \beta_m = \dim(V_0) - \dim(V_1) + \cdots + (-1)^m \dim(V_m).$$

Hence we get a second formula for the Euler characteristic of $(\mathbf{V}, \mathbf{d})$ (see (23)).

A.17(b) Conclude that if $(\mathbf{V}, \mathbf{d})$ is exact, then:

$$\dim(V_0) - \dim(V_1) + \cdots + (-1)^d \dim(V_d) = 0.$$

Appendix B. Exercises Assigned in March 2025

B.1. **Basic Facts About Continuous Maps and Topological Spaces. I will only assign a subset of the problems in this subsection.**

If you have never studied point-set topology, you may want to do more of the exercises in this subsection.

For all problems after this subsection, you can assume the results in this subsection.

Note: Many commonly used functions $\mathbb{R}^n \to \mathbb{R}^m$ are known to be continuous. We will not prove these results from scratch, but we will give some tools to do so in the exercises below. Generally speaking, you can assume that the following functions $\mathbb{R} \to \mathbb{R}$ are continuous: (1) functions $x \mapsto ax+b$ for $a, b \in \mathbb{R}$; (2) $x \mapsto |x|$; $x \mapsto \sqrt{x}$, as a function restricted to $x \geq 0$; (3) common trigonometric functions, such as $x \mapsto \sin(x)$; (4) exponentials, $x \mapsto a^x$ for real $a > 0$; (5) logarithms, $x \mapsto \log_a(x)$ for real $a > 0$; etc. The exercises below give you additional tools to build continuous functions, especially functions $\mathbb{R}^n \to \mathbb{R}^m$ given (1)–(5) above.

Exercise B.1. Let $(X, \mathcal{O})$ be a topological space, and let f, g be continuous maps $X \to \mathbb{R}$ (where $\mathbb{R}$ has its usual topology of open subsets of $\mathbb{R}$). Show that:

B.1(a) $f + g$ is continuous;

B.1(b) fg (their product) is continuous;

B.1(c) if $g(x) \neq 0$ for all $x \in X$, then f/g is continuous.

Exercise B.2. Let $(X,\mathcal{U})$ and $(Y,\mathcal{O})$ be topological spaces.

B.2(a) Show that the identity map $\mathrm{id}_X \colon X \to X$ (i.e., given by $\mathrm{id}_X(x) = x$ is continuous.

B.2(b) If $y_0 \in Y$, show that the map $f \colon X \to Y$ given by $f(x) = y_0$, i.e., the constant map (to y_0), is continuous.

B.2(c) Use the above, plus the previous exercise, to show that if $a, b \in \mathbb{R}$, then the map $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax + b$ is continuous; state very carefully which parts of this exercise and the previous one you are using.

B.2(d) Directly show that the map $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax+b$ is contiuous, by showing that if $U \subset \mathbb{R}$ is open then

$$f^{-1}U \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid ax + b \in U\}$$

is an open set in $\mathbb{R}$.

Exercise B.3. Let $(X_1, \mathcal{O}_1)$, $(X_2, \mathcal{O}_2)$, $(X_3, \mathcal{O}_3)$ be topological spaces. Show that if $f_1 \colon X_1 \to X_2$ is continuous, and $f_2 \colon X_2 \to X_3$ is continuous, then $f_2 \circ f_1 \colon X_1 \to X_3$ is also continuous.

Exercise B.4. Let $(X,\mathcal{U})$ and $(Y,\mathcal{O})$ be topological spaces, and $f \colon X \to Y$ be a continuous map. Let $X' \subset X$ be endowed with topology of relatively open subsets (Definition 9.19). Show that the restriction of f to X', i.e.,

$$f|_{X'} \colon X' \to Y,$$

is continuous.

Exercise B.5. Let $(X,\mathcal{U})$ and $(Y,\mathcal{O})$ be topological spaces, and $f \colon X \to Y$ be a map of sets. Let $Y' \subset Y$ be endowed with topology of relatively open subsets (Definition 9.19). Say that $f(X) \subset Y'$, and let $f' \colon X \to Y'$ given by $f'(x) = f(x)$ (i.e., f' is the same map as f, but now $\mathrm{codomain}(f') = Y'$ ("codomain" is sometimes called the "range"). Show that f is continuous iff f' is continuous.

[To add after 2025: Also if $\sim$ is an equivalence relation on X, and $f \colon X \to Y$ is continuous, then provided that $x_1 \sim x_2$ implies $f(x_1) = f(x_2)$, f extends to a map $X/\sim \to Y$, and this map is also continuous. (Thanks to RY.)]

Exercise B.6. Let $(X,\mathcal{U})$, $(Y_1,\mathcal{O}_1)$, and $(Y_2,\mathcal{O}_2)$ be topological spaces. Let $f_1 \colon X \to Y_1$ and $f_2 \colon X \to Y_2$ be maps of sets, and let $f \colon X \to Y_1 \times Y_2$ be the map of sets.

$$f(x) = \bigl(f_1(x), f_2(x)\bigr).$$

Show that f is continuous iff f_1 and f_2 are continuous. (Recall Definition 9.23.)

Exercise B.7. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be topological spaces with their relative topologies. Let $L \colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.

B.7(a) Show that L is continuous. To prove this, carefully state which of the above exercises you will use.

B.7(b) Let $\mathbf{c} \in \mathbb{R}^m$, and let $M \colon \mathbb{R}^n \to \mathbb{R}^m$ be given by $M(\mathbf{x}) = L(\mathbf{x}) + \mathbf{c}$.

B.2. **Simplicial Complexes and Topological Spaces.**

Exercise B.8. Let $S \subset \mathbb{R}^N$ and $S' \subset \mathbb{R}^{N'}$ be two n-simplices. Hence

$$S = \mathrm{Conv}(\mathbf{a}_0, \dots, \mathbf{a}_n), \quad S' = \mathrm{Conv}(\mathbf{a}'_0, \dots, \mathbf{a}'_n),$$

where $\mathbf{a}_0, \dots, \mathbf{a}_n \in \mathbb{R}^N$ are in general position, and $\mathbf{a}'_0, \dots, \mathbf{a}'_n \in \mathbb{R}^{N'}$ are in general position.

B.8(a) Let $L\colon S \to S'$ be the bijection of sets given by barycentric coordinates,

$$L(t_0\mathbf{a}_0 + \cdots + t_n\mathbf{a}_n) = t_0\mathbf{a}'_0 + \cdots + t_n\mathbf{a}'_n.$$

Show that L is continuous, by quoting results in Subsection B.1.

B.8(b) Using the result from part (a), show that S and S' are homeomorphic.

Exercise B.9. Let K be a simplicial complex in $\mathbb{R}^N$. and K' another in $\mathbb{R}^{N'}$. Let $\mathsf{K}_{\mathsf{abs}}, \mathsf{K}'_{\mathsf{abs}}$ be their associated abstract simplicial complexes, and V, V' their vertex sets. Say that $\mathsf{K}_{\mathsf{abs}}, \mathsf{K}'_{\mathsf{abs}}$ are isomorphic, i.e., there is a bijection $f\colon V \to V'$, such that for each subset $\{v_0, \dots, v_d\} \subset V$ we have

$$\mathrm{Conv}(v_0, \dots, v_d) \in \mathsf{K}_{\mathsf{abs}} \iff \mathrm{Conv}\big(f(v_0), \dots, f(v_d)\big) \in \mathsf{K}'_{\mathsf{abs}}$$

(i.e., f sets up a bijection between the elements of $\mathsf{K}_{\mathsf{abs}}$ and those of $\mathsf{K}'_{\mathsf{abs}}$). Show that $|K| \subset \mathbb{R}^N$ is isomorphic to $|K'| \subset \mathbb{R}^{N'}$.

Exercise B.10. The *discrete topology* on a set, X, refers to the topological space $(X, \mathcal{O})$ where $\mathcal{O}$ is the set of all subsets of X.

B.10(a) Let $\mathbb{N} = \{1, 2, 3, \dots\}$. Show that the relatively open sets of $\mathbb{N}$ as a subset of $\mathbb{R}$ consists of all subsets of $\mathbb{N}$. This is therefore the discrete topology on $\mathbb{N}$.

B.10(b) Let $S = \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$. Show that the relatively open sets of S as a subset of $\mathbb{R}$ does not consist of all subsets of S, i.e., find a subset of S that is not relatively open.

Exercise B.11. Let $\mathbb{Q} \subset \mathbb{R}$ be the set of rational numbers. Let $\sim$ be the equivalence relation on $\mathbb{R}$ given by $x_1 - x_2 \in \mathbb{Q}$.

B.11(a) Show that $\sim$ is an equivalence relation.

B.11(b) Show that the only open subsets of $\mathbb{R}/\sim$ are $\emptyset$ and all of $\mathbb{R}/\sim$.

Exercise B.12. Let $A \subset X$. We use X/A to denote the quotient space $X/\sim$ where $x_1 \sim x_2$ iff $x_1 = x_2$ or $x_1, x_2 \in A$. Let $A = \{0, 1\}$ and $X = [0, 1]$; show that X/A is isomorphic to $\mathbb{S}^1$. [Recall: you can assume that trigonometric functions are continuous.]

Exercise B.13. Let $\mathsf{K}', \mathsf{K}''$ be abstract simplicial complexes, and v', v'' a vertex of each. We use $\mathsf{K} = \mathsf{K}' \vee \mathsf{K}''$ to denote the abstract simplicial complex $\mathsf{K}' \amalg \mathsf{K}''$, where we identify v' and v'' (hence $\mathsf{K} = \mathsf{K}' \vee \mathsf{K}''$ generally depends on the choice of v', v''). [Hence K can be described as follows: there is a vertex v of K representing v', v'', and a simplex $S \in \mathsf{K}$ is either of the form: (1) $S' \in \mathsf{K}'$, where if v' appears in S' then v' is replaced with v; or (2) $S'' \in \mathsf{K}''$, similarly replacing v'' with v if it appears in S''.

B.13(a) Use a Mayer-Vietoris sequence to show that if $\mathsf{K}', \mathsf{K}''$ is connected, then ($\mathsf{K} = \mathsf{K}' \vee \mathsf{K}''$ is connected and) for $i \geq 1$ we have

$$H_i(\mathsf{K}) = H_i(\mathsf{K}') \oplus H_i(\mathsf{K}'').$$

B.13(b) Show that for any sequence of non-negative integers $m_0, \ldots, m_k$ with $m_0 \geq 1$, there is an abstract simplicial complex K whose Betti numbers are: $\beta_i(\mathsf{K}) = m_i$ for $0 \leq i \leq k$, and $\beta_i(\mathsf{K}) = 0$ for $i \geq k+1$. [Hint: you may use the fact that for any set, S, of size $|S| = n \geq 2$, we have that $\mathsf{K} = \mathrm{Power}(S) \setminus S$ has $\beta_i(\mathsf{K}) = 1$ for $i = 0, n-1$, and otherwise $\beta_i(\mathsf{K}) = 0$.]

Exercise B.14. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be given their induced subset topologies. For $x_0 \in X$ and $y_0 \in Y$, we define $X \vee Y \subset \mathbb{R}^{n+m}$ to be the subset

$$X \vee Y \stackrel{\text{def}}{=} \{(x, y_0) | x \in X\} \cup \{(x_0, y) | y \in Y\} \tag{78}$$

(which therefore depends on x_0 and y_0). Let K' and K'' be simplicial complexes in, respectively $\mathbb{R}^n$ and $\mathbb{R}^m$, and let $\mathsf{K}', \mathsf{K}''$ be their associated abstract simplicial complexes. [Some extra text was added for clarification] Then $|K| \subset \mathbb{R}^n$ and $|K'| \subset \mathbb{R}^m$, and so (78) defines $|K| \vee |K'|$ as a subset of $\mathbb{R}^{n+m}$.

B.14(a) Show that the topological space Z/A, where $Z = X \amalg Y$ (where $\amalg$ is the disjoint union) and $A = \{x_0, y_0\}$, is homeomorphic to $X \vee Y$ above. Hence $X \vee Y$ can be more simply defined as a topological space arising from X and Y.

B.14(b) Describe a simplicial complex, L, in $\mathbb{R}^{n+m}$ such that $|L| = |K'| \vee |K''|$.

B.14(c) Show that if L is the abstract simplicial complex associated to L, then $\mathsf{L} = \mathsf{K}' \vee \mathsf{K}''$ (with notation as in the previous problem), where $\vee$ identifies the vertex x of K' with y of K''.

Exercise B.15. Let $X \subset \mathbb{R}^n$, and let f be as in (48). Show that f is continuous.

Exercise B.16. Let $X = (0,1) \subset \mathbb{R}$ (with its subspace topology), and let f be as in (48). Show that f^{-1} is not continuous.

Exercise B.17. A topological space is *compact* if any open covering of X, i.e., any family of open subsets $\{U_i\}_{i \in I}$ with $\cup_{i \in I} U_i = X$, has a finite subcovering, i.e., for some finite set $I' \subset I$ we also have $\cup_{i \in I'} U_i = X$. Show that if $X \subset \mathbb{R}^n$ is compact, then with f as in (48) we have that f^{-1} is continuous.

Exercise B.18. If X is a topological space and $A \subset X$, a *weak deformation retraction from X to A* is a map $F \colon X \times [0,1] \to X$ such that $F(\cdot, 0) \colon X \to X$ is the identity map, and $F(\cdot, 1) \colon X \to X$ is a retraction to A (Definition 9.47). A *strong deformation retraction* adds the condition that for all $0 \leq t \leq 1$, $F(\cdot, t)|_A$ is the identity on A. [Hatcher [Hat02] use "deformation retraction" to mean a strong deformation retraction; other authors (e.g., Massey [Mas80] page 21, Definition 4.4, Fulton [Ful95], page 93, just below Proposition 6.23) use "deformation retraction" to mean a weak deformation retraction.] Show that in either case (i.e., if there is a weak deformation retraction from X to A), then X and A are homotopy equivalent (recall the definition at the end of Subsection 9.8, just before the start of Subsection 9.9).

Exercise B.19. We say that X is *weakly (strongly) collapsible* if there is a weak (strong) retraction from X to a point $x \in X$. (See the previous problem for definitions.) Write out these conditions in simpler terms.

B.3. **Applications of the Brouwer Fixed Point Theorem: The Perron-Frobenius Theorem, Nash Equilibrium, etc. In March 2025, we will do some parts of the exercises below in class. You don't have to submit Exercise B.23, but you should understand it since it is a special case of (and a "warm-up" for) Exercise B.24. The exercises in this subsection are all applications of the Brouwer fixed point theorem, Theorem 9.46, or related to its application. You do not need to submit Exercise B.20, since it uses compactness, which we did not cover in 2025. Thanks to office hours with QW on March 27, you don't need compactness to do Exercise B.20; but if you don't, the argument is a more elaborate computation.**

Exercise B.20. The point of this exercise is to give many examples of $X \subset \mathbb{R}^n$ that are homeomorphic to $\mathbb{D}^n$. Say that $X \subset \mathbb{R}^n$ is *star-shaped at* x_0 if (1) x_0 lies in the interior of X, (2) X is closed and bounded (i.e., X is compact), and (3) for each $\mathbf{x} \neq \mathbf{0}$, there is a unique $t > 0$ such that $\mathbf{x}_0 + t\mathbf{x} \in \partial X$, where ∂X is the boundary of X (i.e., X minus its interior).[23]

B.20(a) Let $\mathbf{x} \neq \mathbf{0}$, and let $\mathbf{f}(\mathbf{x})$ be the unique positive number such that $\mathbf{x}_0 + f(\mathbf{x})\mathbf{x} \in \partial X$. Show that $f\colon X \to \mathbb{R}_{\geq 0}$ is continuous. (Here $\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers.) [Hint: look at the proof of the Brouwer fixed point theorem, specifically the argument that shows that g is contiuous; in more detail, for any $\mathbf{x} \neq \mathbf{0}$ and any $\delta_1 > 0$ we have that $\mathbf{x}_0 + (f(\mathbf{x}) + \delta_1)\mathbf{x}$ is in $\mathbb{R}^n \setminus X$, which is an open set, and $\mathbf{x}_0 + (f(\mathbf{x}) - \delta_1)\mathbf{x}$ lies in $X \setminus \partial X$, i.e., the interior of X, which is also an open set.]

B.20(b) Define the map $g\colon \mathbb{D}^n \to X$ where for each $|\mathbf{x}| = 1$ and $0 \leq t \leq 1$,

$$g(t\mathbf{x}) \stackrel{\text{def}}{=} \mathbf{x}_0 + tf(\mathbf{x})\mathbf{x}.$$

Corrected on March 27 in office hours by QW. Show that g is a homeomorphism.

B.20(c) Show that the restriction of g to $\mathbb{S}^{n-1}$ gives a homeomorphism from $\mathbb{S}^{n-1} \to \partial X$.

Exercise B.21. Let M be an $n \times n$ matrix with non-negative, real entries, and $n \geq 1$. We say that M is *irreducible* if for all $i, j \in [n]$, for some $k \in \mathbb{N} = \{1, 2, \ldots\}$, $(M^k)_{ij}$, i.e., the $(i, j) - th$ entry of M^k, is positive.

B.21(a) Let M be any $n \times n$ matrix with non-negative, real entries such that each column (not row) of M has at least one non-zero entry. Show that for some $\mathbf{x} \in \Delta^{n-1} = \mathrm{Conv}(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ we have $M\mathbf{x} = \lambda\mathbf{x}$ for some $\lambda > 0$. [Hint: consider the function $g\colon \Delta^{n-1} \to \Delta^{n-1}$ given by $g(\mathbf{x}) = M\mathbf{x}/((M\mathbf{x}) \cdot (\mathbf{1}))$, where $\mathbf{1} = (1, \ldots, 1)$ and $\cdot$ is the usual "dot product."]

[23] Munkres' textbook [Mun84] defines a weaker notion of *star-convex* for an open, bounded subset $U \in \mathbb{R}^n$, namely that for any $\mathbf{x} \in U$, $\mathrm{Conv}(\mathbf{0}, \mathbf{x}) \subset U$; see Exercise 5 there, end of Section 1, page 7. For this weaker definition, $\overline{U}$, i.e., the closure of U, need not be star-shaped at $\mathbf{0}$.

B.21(b) Show that if, in addition, M is irreducible, then each entry of $\mathbf{x}$ is positive. [In this case, λ is called the *Perron-Frobenius eigenvalue* of M, and $\mathbf{x}$ is called a Perron-Frobenius eigenvector.]

B.21(c) Show that in the previous part, if $M\mathbf{y} = \mu\mathbf{y}$ for some $\mu \in \mathbb{C}$ and non-zero vector $\mathbf{y} \in \mathbb{C}^n$ (i.e., $(\mu, \mathbf{y})$ is a pair of an eigenvalue and a corresponding eigenvector), then $|\mu| \le \lambda$ [Hint: consider the maximum value, $\tilde{m}$ (the original version used M, which is a clash of notation), of $|y_i|/x_i$ (where $\mathbf{y} = (y_1, \ldots, y_n)$ and similarly for $\mathbf{x}$); hence $|y_j| \le \tilde{m}x_j$ and equality holds when $j = i$; apply absolute values to the equation $\mu y_i = \sum_j M_{ij} y_j$.]

B.21(d) Show that in the previous part, if $\mu = \lambda$, then $\mathbf{y} = \alpha\mathbf{x}$ for some nonzero $\alpha \in \mathbb{C}$. [Hint: again, we can assume $|y_j| \le \tilde{m}x_j$ for all j, and that equality holds for $j = i$; let α be such that $y_i = \alpha x_i$; applying absolute values to both sides of $\mu y_i = \sum_j M_{ij} y_j$, show that $y_j = \alpha x_j$ for all j such that $M_{ij} > 0$.]

B.21(e) Show that in part (c), if $k \in \mathbb{N}$ and M^k has a nonzero diagonal element, and if $|\mu| = \lambda$, then $\mu = \lambda\zeta$ where $\zeta^k = 1$.

We remark that the GCD (greatest common divisor), p, of all k such that M^k has a nonzero diagonal element is called the *period of* M. It is not hard to show that (1) for sufficiently large $n \in \mathbb{N}$, all diagonal elements of M^{np} are nonzero, and (2) $[n]$ can be partitioned into subsets $V_0, \ldots, V_{p-1}$ such that if $M_{ij} \neq 0$, then $i \in V_r$ implies that $j \in V_{r+1 \bmod p}$.

Exercise B.22. We say that an $n \times n$ matrix, P, with non-negative real entries is a Markov matrix if each row of P is stochastic, i.e., if p_{ij} is the (i, j)-th entry of P, then for all i we have

$$\sum_{j=1}^{n} p_{ij} = 1.$$

Warning: in probability theory, a Markov matrix generally acts to the right of a row vector; so when we write $\mathbf{a}P$, we understand $\mathbf{a}$ to be a row vector.

B.22(a) Show that $P\mathbf{1} = \mathbf{1}$. Note: since P is acting to the left of $\mathbf{1}$, we understand that $\mathbf{1}$ is a column vector.

B.22(b) Show that if $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ is any stochastic row vector (i.e., $\pi_1, \ldots, \pi_n$ are non-negative and sum to 1), then $(\boldsymbol{\pi}P) \cdot \mathbf{1} = 1$, and that $\boldsymbol{\pi}P$ is a stochastic (row) vector.

B.22(c) Say that P is *irreducible*, i.e., for all $i, j \in [n]$ there is an integer $k \ge 0$ such that $(P^k)_{ij} > 0$ (the i, j entry of P^k). Show that there is a unique stochastic (red) vector $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ such that $\boldsymbol{\pi} = \boldsymbol{\pi}P$. (You may use the results of the previous exercise.).

Exercise B.23. Say that there are $a, b \in \mathbb{R}$ such that for each $0 \le p \le 1$, we define

$$\text{Reward}(p) = ap + b(1 - p).$$

[This "Reward" is just a definition, but we intuitively think of a "game" where you choose a value $p \in [0, 1]$, and you are "rewarded" or "paid" the value of Reward(p); you can also think of a and b as being the "payouts" of playing one of two "pure" strategies, and Reward(p) being the payout of a "mixed strategy" of playing one strategy with probability p, the other with probability $1-p$.] Fix a real p with $0 \le p \le 1$, and set

$$\mathrm{RewardToSwitch}_1 \stackrel{\mathrm{def}}{=} \max\bigl(0, \mathrm{Reward}(1) - \mathrm{Reward}(p)\bigr),$$

$$\mathrm{RewardToSwitch}_0 \stackrel{\mathrm{def}}{=} \max\bigl(0, \mathrm{Reward}(0) - \mathrm{Reward}(p)\bigr),$$

which therefore represents the rewards to switching from a given p to $p = 1$ and $p = 0$ respectively, i.e., to the two mixed strategies; let

$$\mathbf{RewardToSwitch} = \bigl(\mathrm{RewardToSwitch}_1, \mathrm{RewardToSwitch}_0\bigr)$$

which organizes the RewardToSwitch values into a vector. Say that for some c we have

$$\mathbf{RewardToSwitch} = c\,(p, 1-p).$$

B.23(a) Show that if $0 < p < 1$ then $a = b$.

B.23(b) Show that if $p = 0$ then $a \le b$.

B.23(c) Show that if $p = 1$ then $a \ge b$.

B.23(d) Show that whatever p is, i.e., in all cases above, $c = 0$, i.e.,

$$\mathbf{RewardToSwitch} = \mathbf{0} = (0, 0),$$

and for any $q \in [0, 1]$ we have

$$\mathrm{Reward}(q) \le \mathrm{Reward}(p).$$

B.23(e) For each $\mathbf{x} \in \mathbb{R}^2$ with $x_1, x_2 \ge 0$ and $\mathbf{x} \ne (0, 0)$, define

$$\mathrm{Stochastic}(\mathbf{x}) = \frac{\mathbf{x}}{\mathbf{x} \cdot \mathbf{1}} = \frac{\mathbf{x}}{x_1 + x_2}.$$

Show that for any $\mathbf{p} \in \Delta^1 = \mathrm{Conv}(\mathbf{e}_1, \mathbf{e}_2) \subset \mathbb{R}^2$ and any $\mathbf{R} = (R_1, R_2)$ with $R_1, R_2 \ge 0$ we have

$$\mathrm{Stochastic}(\mathbf{p}, \mathbf{R}) = \mathbf{p}$$

iff $\mathbf{R} = c\,\mathbf{p}$ for some c. [The point of this part is that it is an idea used by Nash in his original proof of the existence of a *Nash equilibrium*, to reduce its existence to the Brouwer fixed point theorem; see below.]

Exercise B.24. Let $\mathbf{a} \in \mathbb{R}^n$, and for each $\mathbf{p} \in \Delta^{n-1} = \mathrm{Conv}(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ we define

$$\mathrm{Reward}(\mathbf{p}) \stackrel{\mathrm{def}}{=} \mathbf{a} \cdot \mathbf{p} = a_1 p_1 + \cdots + a_n p_n,$$

(which we interpret as a "reward" or "payout" of the "mixed strategy" $\mathbf{p}$ of "playing strategy i" with probability p_i), and

$$\mathrm{RewardToSwitch}_i(\mathbf{p}) \stackrel{\mathrm{def}}{=} \bigl(\max(0, \mathrm{Reward}(\mathbf{e}_i) - \mathrm{Reward}(\mathbf{p})\bigr) = \bigl(\max(0, a_i - \mathrm{Reward}(\mathbf{p})\bigr),$$

(which is therefore the reward to switch to the "pure strategy" $\mathbf{e}_i$), and

$$\mathbf{RewardToSwitch}(\mathbf{p}) = \bigl(\mathrm{RewardToSwitch}_1(\mathbf{p}), \ldots, \mathrm{RewardToSwitch}_n(\mathbf{p})\bigr),$$

which therefore gathers the rewards to switch into a vector. Say that for some $c \in \mathbb{R}$ and $\mathbf{p} \in \Delta^{n-1}$ we have

$$\mathbf{RewardToSwitch}(\mathbf{p}) = c\,\mathbf{p}.$$

For the exercises below, you can use the results of Exercise B.23 (or not...).

B.24(a) Show that if $p_i, p_j > 0$, then $a_i = a_j$.

B.24(b) Show that if $p_i > 0$ and $p_j = 0$, then $a_i \ge a_j$.

B.24(c) Show that $c = 0$, i.e.,

$$\mathrm{RewardToSwitch}(\mathbf{p}) = \mathbf{0} = (0, \ldots, 0),$$

and that for any $\mathbf{q} \in \Delta^{n-1}$ we have

$$\mathrm{Reward}(\mathbf{q}) \le \mathrm{Reward}(\mathbf{p}).$$

B.24(d) For each $\mathbf{x} \in \mathbb{R}^n$ with non-negative coefficients and $\mathbf{x} \ne \mathbf{0}$, let

$$\mathrm{Stochastic}(\mathbf{x}) = \frac{\mathbf{x}}{\mathbf{x} \cdot 1} = \frac{\mathbf{x}}{x_1 + \cdots + x_n}, \tag{79}$$

which therefore lies in Δ^{n-1}. Show that for any $\mathbf{p} \in \Delta^{n-1}$ and $\mathbf{R} \in \mathbb{R}^n$ with non-negative coefficients,

$$\mathrm{Stochastic}(\mathbf{p} + \mathbf{R}) = \mathbf{p}$$

iff $\mathbf{R} = c\mathbf{p}$ for some $c \ge 0$. [The point of this part is that it is an idea used by Nash in his original proof of the existence of a *Nash equilibrium*, to reduce its existence to the Brouwer fixed point theorem; see below.]

B.24(e) Let $I \subset [n]$ be the set of i where a_i attains its maximum value. What is the set of $\mathbf{p} \in \Delta^{n-1}$ where $\mathrm{Reward}(\mathbf{p})$ attains its maximum value?

Exercise B.25. In this exercise we prove the existence of a *Nash equilibrium* for a k-player game with a finite set of "pure strategies" for each player; see [Nas51]. You may use the results in Exercise B.24. Let $k \in \mathbb{N} = \{1, 2, \ldots\}$. By a *$k$-person game* we mean any function

$$A \colon [k] \times [n_1] \times \cdots \times [n_k] \to \mathbb{R}. \tag{80}$$

where $(n_1, \ldots, n_k) \in \mathbb{N}^k$. We interpret A is a "k-person game," where each $j_m \in [n_m]$ represents a ("pure") "strategy" that player $j \in [k]$ can play, and $A(i; j_1, \ldots, j_k)$ is the "reward" or "payout" to person i, when each player $\ell \in [k]$ plays strategy j_ℓ. Let $\boldsymbol{\Delta}$ denote $\Delta^{n_1-1} \times \ldots \times \Delta^{n_k-1}$. For any sequence $\mathfrak{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_k) \in \boldsymbol{\Delta}$, and any $i \in [k]$, we define

$$\mathrm{Reward}_i(\mathfrak{p}) \stackrel{\mathrm{def}}{=} \sum_{(j_1,\ldots,j_k) \in [n_1] \times \ldots [n_k]} A(i; j_1, \ldots, j_k) \mathbf{p}_1(j_1) \ldots \mathbf{p}_k(j_k). \tag{81}$$

So we interpret the above as the "reward" or "payout" to person i when each player $\ell \in [k]$ plays a "mixed strategy" $\mathbf{p}_\ell$. For each such i and $\mathfrak{p}$, and each $m \in [n_i]$ we define

$$\mathrm{RewardToSwitch}_{i,m}(\mathfrak{p}) \stackrel{\mathrm{def}}{=} \bigl(\max(0, \mathrm{Reward}_i(\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{e}_m, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_k) - \mathrm{Reward}_i(\mathfrak{p})\bigr),$$

which is therefore the maximum of 0 and the reward to player i for switching their strategy from $\mathbf{p}_i$ to $\mathbf{e}_m$, with the strategies of all other players fixed. We say that $\mathfrak{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_k)$ is a *Nash equilibrium* if $\mathrm{RewardToSwitch}_{i,m}(\mathfrak{p}) = 0$ for all i and $m \in [n_i]$. We define the "i-th Nash reward vector of $\mathfrak{p}$" to be

$$\mathbf{RewardToSwitch}_i(\mathfrak{p}) = \bigl(\mathrm{RewardToSwitch}_{i,1}(\mathfrak{p}), \ldots, \mathrm{RewardToSwitch}_{i,n_i}(\mathfrak{p})\bigr),$$

which therefore combines the rewards to switch for player i into a vector of rewards. For any $i \in [k]$ and $\mathfrak{p} \in \boldsymbol{\Delta}$ define the "Nash modified i-player strategy" by

$$\mathrm{NashModified}_i(\mathfrak{p}) = \mathrm{Stochastic}\bigl(\mathbf{p}_i + \mathbf{RewardToSwitch}_i(\mathfrak{p})\bigr) \tag{82}$$

with Stochastic as in (79), and let

(83) $$\mathbf{NashModified}(\mathfrak{p}) = \big(\mathrm{NashModified}_1(\mathfrak{p}), \ldots, \mathrm{NashModified}_k(\mathfrak{p})\big).$$

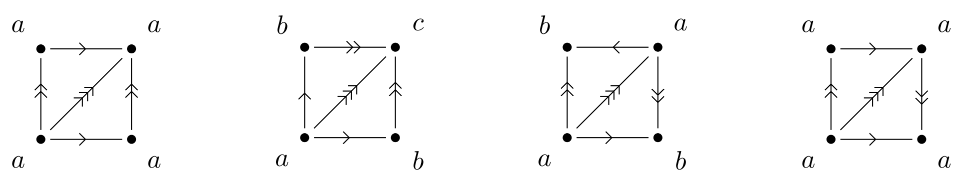
B.25(a) Show that NashModified: $\mathbf{\Delta} \to \mathbf{\Delta}$ is continuous.
B.25(b) Show that $\mathbf{\Delta}$ is homeomorphic to the disk $\mathbb{D}^N$ where $N = n_1 + \ldots + n_k$. Conclude that NashModified has a fixed point.
B.25(c) Let $\mathfrak{p}$ be a fixed point of NashModified. Show that $\mathfrak{p}$ is a Nash equilibrium. [Hint: Use the results of Exercise B.24.]
B.25(d) Say that for a real $\alpha > 0$ you define

$$\mathrm{NashModified}_\alpha(\mathfrak{p}) = \mathrm{Stochastic}\big(\mathbf{p}_i + \alpha\mathbf{RewardToSwitch}_i(\mathfrak{p})\big).$$

Show that a fixed point of $\mathrm{NashModified}_\alpha$ is also a Nash equilibrium. [Hence the definition Nash's modified strategy allows for some flexibility. For much greater flexibility, see Exericse C.7.]

B.4. Δ-Complex Computations.

Exercise B.26. Consider the following Δ-complexes:



Torus $\qquad$ $\mathbb{S}^2$ $\qquad$ $\mathbb{RP}^2$ $\qquad$ Klein Bottle

For the torus, we computed its simplicial homology (as a Δ-complex) in class on March 24. By a similar computation on these other Δ-complexes, compute the homology groups of:

B.26(a) $\mathbb{S}^2$ (this is two 2-simplices glued together on their boundary, which you can easily see is homeomorphic to the suspension of $\mathbb{S}^1$) (of course, we have already computed the homology groups of $\mathbb{S}^2$ by other means);
B.26(b) $\mathbb{RP}^2$ (you should be able to convince yourself that this is isomorphic to the $\mathbb{RP}^2$ defined in class, i.e., $\mathbb{S}^2/\sim$ where $\sim$ identifies each point $\mathbf{x}$ with $-\mathbf{x}$); and
B.26(c) the Klein Bottle (which is basically a cylinder $[0,1] \times \mathbb{S}^1$ with its bounding circles identified in "opposite orientation").

B.5. Barcodes.

Exercise B.27. Say that $\mathsf{K}^0 \subset \mathsf{K}^1 \subset \cdots \subset \mathsf{K}^m$ are an increasing sequence of simplicial complexes that all have the same set, V, of vertices (i.e., 0-simplicies, i.e., sets of size 1). Say also that $\mathsf{K}^0 = V \cup \{\emptyset\}$ (hence K^0 consists of V, each an isolated point of K^0, so that

$$\beta_0(\mathsf{K}^0) = \dim\big(H_0(\mathsf{K}^0)\big) = |V|.$$

We easily see that $\beta_0(\mathsf{K}^i)$ is non-increasing in i, and $\beta_0(\mathsf{K}^i) = 1$ provided that V is connected in K^i. Hence the barcode decomposition of $H_0(\mathsf{K}^i)$ is particularly simple, and the groups $H_0(\mathsf{K}^i)$ depend only on the vertices and edges of K^i. Give a barcode decomposition of $H_0(\mathsf{K}_i)$ in the following cases:

B.27(a) $V = \{v_1, \ldots, v_4\}$, and the edges in each K_i consists of

$$E(\mathsf{K}^0) = \emptyset,$$
$$E(\mathsf{K}^1) = \bigl\{\{v_1, v_2\}\bigr\},$$
$$E(\mathsf{K}^2) = \bigl\{\{v_1, v_2\},\ \{v_2, v_3\}\bigr\},$$
$$E(\mathsf{K}^3) = \bigl\{\{v_1, v_2\},\ \{v_2, v_3\},\ \{v_3, v_4\}\bigr\}.$$

B.27(b) $V = \{v_1, \ldots, v_4\}$, and the edges in each K_i consists of

$$E(\mathsf{K}^0) = \emptyset,$$
$$E(\mathsf{K}^1) = \bigl\{\{v_1, v_2\}\bigr\},$$
$$E(\mathsf{K}^2) = \bigl\{\{v_1, v_2\},\ \{v_3, v_4\}\bigr\},$$
$$E(\mathsf{K}^3) = \bigl\{\{v_1, v_2\},\ \{v_2, v_3\},\ \{v_3, v_4\}\bigr\}.$$

B.6. **More on Relative Homology.** Recall from Subsection 9.10, if $A \subset X$ is a subset of a topological space, then we define an *i-singular chain of X relative to A* to be an element of

$$\mathcal{C}_i^{\rm sing}(X)/\mathcal{C}_i^{\rm sing}(A),$$

and the maps ∂_i on $\mathcal{C}_i^{\rm sing}(X)$ and on $\mathcal{C}_i^{\rm sing}(A)$ give rise to maps

$$\cdots \xrightarrow{\partial_3} \mathcal{C}_2^{\rm sing}(X,A) \xrightarrow{\partial_2} \mathcal{C}_1^{\rm sing}(X,A) \xrightarrow{\partial_1} \mathcal{C}_0^{\rm sing}(X,A) \to 0$$

which form a chain complex (i.e., $\partial_i\partial_{i+1} = 0$ for all integers $i \ge 0$). We define $H_i^{\rm sing}(X,A)$ to be the homology groups of this sequence. We easily see that

$$0 \to \mathcal{C}_i^{\rm sing}(A) \to \mathcal{C}_i^{\rm sing}(X) \to \mathcal{C}_i^{\rm sing}(X,A) \to 0$$

is short exact for all i, and this yields a long exact sequence

$$\cdots \to H_i^{\rm sing}(A) \to H_i^{\rm sing}(X) \to H_i^{\rm sing}(X,A) \to H_{i-1}^{\rm sing}(A) \to \cdots$$

MORE TO COME...

Appendix C. Exercises Not Assigned in Spring 2025

These exercises will not be assigned in Spring 2025, but they may be interesting.

Exercise C.1. For any integer $n \ge 0$, the *n-sphere*, $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is the set

$$\mathbb{S}^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

C.1(a) Show that any three points on $\mathbb{S}^1$ are in general position.
C.1(b) Are any four points on $\mathbb{S}^2$ in general position?

Exercise C.2. Let $U \subset W$ be finite dimensional vector spaces over a field, $\mathbb{F}$. Let $B = \{u_1, \ldots, u_m\}$ be a basis for U. By the *basis exchange theorem*[24], there is a basis for W consisting of B plus some additional vectors $w_1, \ldots, w_p$.

C.2(a) Show that $w_1 + U, \ldots, w_p + U$ is a basis for W/U.
C.2(b) Conclude that $\dim(W/U) = \dim(W) - \dim(U)$.

[24] If you don't know this theorem, you should look it up in a linear algebra textbook; at UBC, we tend to use the textbooks [J$\ddot{9}$4, Axl15] (currently free of charge to UBC Library card holders).

Exercise C.3. Let $U_1, \dots, U_r \subset U$ be $\mathbb{F}$-vector spaces for some field, $\mathbb{F}$, and say that U equals the internal direct sum $U_1 \oplus \cdots \oplus U_r$ (hence every vector in U can be written uniquely as a sum of vectors in $U_1, \dots, U_r$.

C.3(a) Show that if for each $i = 1, \dots, r$, $\mathcal{B}_i$ is a basis for U_i, then:
  (a) the sets $\mathcal{B}_1, \dots, \mathcal{B}_r$ are disjoint;
  (b) the set $\mathcal{B} = \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_r$ are linearly independent; and
  (c) each vector in U can be written uniquely as a linear combination of the vectors in $\mathcal{B}$.

C.3(b) Explain why, using the above, we have

$$\dim(U) = \sum_{i=1}^{r} \dim(U_i).$$

Exercise C.4. Let $U_1, \dots, U_r \subset U$ be $\mathbb{F}$-vector spaces for some field, $\mathbb{F}$, and say that U equals the internal direct sum $U_1 \oplus \cdots \oplus U_r$ (hence every vector in U can be written uniquely as a sum of vectors in $U_1, \dots, U_r$. Let $A \subset U$.

C.4(a) Show that if $A_i = A \cap U_i$ and $\mathcal{A}_i$ is a basis for A_i (for $i = 1, \dots, r$), then the $\mathcal{A}_i$ are distinct and their union is linearly independent.

C.4(b) Show that $\dim(A) \geq \sum_{i=1}^{r} \dim(A_i)$.

C.4(c) TO BE CONTINUED...

Exercise C.5. Let

$$0 \to V_2 \xrightarrow{d_2} V_1 \xrightarrow{d_1} V_0 \to 0$$

be a short exact sequence of vector spaces (Definition 5.6).

C.5(a) Show that d_2 is injective.

C.5(b) Show that d_1 is surjective.

C.5(c) Show that if V_0, V_1, V_2 are finite dimensional, then

$$\dim(V_1) = \dim(V_2) + \dim(V_0)$$

using (84).

Exercise C.6. Let $U_1, U_2 \subset U$ be subspaces of an $\mathbb{R}$-vector space U, and $U_1 + U_2$ is their span. Describe a short exact sequence

$$0 \to U_1 \cap U_2 \xrightarrow{\mu} U_1 \oplus U_2 \xrightarrow{\nu} U_1 + U_2 \to 0,$$

constructed similarly to (24).

Exercise C.7. What we called the NashModified in Exercise B.25 works in very great generality. Specifically, consider the one player, n-strategies game in Exercise B.24: hence we fix real numbers $a_1, \dots, a_n$, and for each $\mathbf{p} \in \Delta^{n-1} = \mathrm{Conv}(\mathbf{e}_1, \dots, \mathbf{e}_n)$ we define

$$\mathrm{Reward}(\mathbf{p}) \stackrel{\mathrm{def}}{=} \mathbf{a} \cdot \mathbf{p} = a_1 p_1 + \cdots + a_n p_n.$$

Consider a function $\mathbf{RewardToSwitch}\colon \Delta^{n-1} \to \mathbb{R}^n_{\geq 0}$, where $\mathrm{RewardToSwitch}_i$ denotes i-th component of $\mathbf{RewardToSwitch}$ (hence a function $\Delta^{n-1} \to \mathbb{R}_{\geq 0}$). Say that $\mathbf{RewardToSwitch}$ is *sensible* if for all $i \in [n]$,

$$a_i \leq \mathrm{Reward}(\mathbf{p}) \iff \mathrm{RewardToSwitch}_i(\mathbf{p}) = 0,$$

(i.e., the "reward to switch to i" is positive iff a_i is greater than the current reward).

C.7(a) Say that we have a sensible **RewardToSwitch** function, and for some $c \in \mathbb{R}$ and $\mathbf{p} \in \Delta^{n-1}$ we have

$$\mathbf{RewardToSwitch}(\mathbf{p}) = c\,\mathbf{p},$$

Show that $c = 0$. [Hint: among all i with $p_i > 0$, consider the i at which a_i is minimized.]

C.7(b)* (This part is more difficult only because you need to be familiar with some point-set topology; in particular, you need to know what is a limit point of a sequence of points in $\mathbb{R}^n$.) Say that **RewardToSwitch** is both sensible and continuous. Let $\mathbf{p}^0 \in \Delta^{n-1}$ be an arbitrary vector, and let $\mathbf{p}^1, \mathbf{p}^2, \ldots$ be the sequence given by:

$$\mathbf{p}^{m+1} = \mathrm{Stochastic}\bigl(\mathbf{p}^m + \mathbf{RewardToSwitch}(\mathbf{p}^m)\bigr).$$

Show that if $\mathbf{p}$ is any limit point of the sequence $\{\mathbf{p}^m\}$, then

$$\mathbf{p} = \mathrm{Stochastic}\bigl(\mathbf{p} + \mathbf{RewardToSwitch}(\mathbf{p})\bigr)$$

and that for any $i \in [n]$ such that $a_i < \max_{j\in[n]} a_j$, $p_i = 0$.

C.7(c)* (Here you need to know that any sequence in closed and bounded subset of $\mathbb{R}^n$ has a limit point.) Let p_i^m to denote the i-th component of $\mathbf{p}^m$. Show that for any $i \in [n]$ such that $a_i < \max_{j\in[n]} a_j$,

$$\lim_{m\to\infty} p_i^m = 0.$$

C.7(d)* Say that you have an k-player game, (80), with reward function (81), and for $i \in [k]$ you have a function

$$\mathbf{RewardToSwitch}_i \colon \Delta^{n_1-1} \times \Delta^{n_k-1} \to \Delta^{n_i-1}$$

that is continous and that for any fixed $\mathbf{p}_j$ for all $j \neq i$, the resulting function $\Delta^{n_i-1} \to \mathbb{R}^n_{\geq 0}$ is sensible. Prove Nash's equilibrium existence theorem (i.e., reprove (a)–(c) of Exercise B.25) with these $\mathbf{RewardToSwitch}_i$ functions (so Nash's modified strategy is given in (83) and (82)).

Appendix D. Review of Linear Algebra

In this appendix we summarize the facts we need from linear algebra we need in these notes.

D.1. **"Abstract" Linear Algebra at UBC as of 2025.** Most of the linear algebra we will need is taught at UBC in Math 223 (Honours Linear Algebra), which introduces vector spaces. Sometimes vector spaces are called "abstract vector spaces," to contrast with $\mathbb{R}^n$ and subspaces of $\mathbb{R}^n$ which are sometimes called "concrete vector spaces").[25] Math 223 typically uses textbooks such as [J9̈4, Axl15], currently available online for free to folks with UBC Library privileges.

Everything we need in linear algebra is contained in the (deservedly) classic reference *Matrix Analysis* by Horn and Johnson ([HJ13], or earlier editions, [HJ85, HJ90], and one of these is currently available online for UBC Library clients). The

[25]The more commonly taken course is Math 221 (Linear Algebra), which deals with more "concrete" linear algebra, meaning solving systems of equations, and dealing with $\mathbb{R}^n$ and subspaces thereof (e.g., the kernel or nullspace of a matrix). Math 221 typically has a free online textbook, such as `https://personal.math.ubc.ca/~tbjw/ila/index.html`. Engineering students typically take UBC's Math 152 instead.

basics definitions are in Chapter 0 there. A number of Computer Science courses at UBC use this book as a textbook or reference.

D.2. **Fields.** Linear algebra works over an arbitrary *field*, $\mathbb{F}$, although for most of these notes we work over the field $\mathbb{F} = \mathbb{R}$, the real numbers. We assume you will look up the definition of a *field*;[26] intuitively speaking, a field is a what one calls the "scalars" of a vector space, and these scalars have a $+$ and $\times$ operations (and their inverses, subtraction and, for non-zero elements, division) so that one can "do linear algebra" (e.g., solve systems of linear equations by reducing a matrix to row echelon form) just as one does for $\mathbb{R}$ or $\mathbb{Q}$, the rational numbers.

We often work with the field $\mathbb{R}$, although other useful fields are $\mathbb{Q}$ (the rational numbers), $\mathbb{C}$ (the complex numbers), and finite fields (for any prime p, $\mathbb{Z}/p\mathbb{Z}$ is a field, although for any $m \in \mathbb{N}$ there is an essentially unique field with p^m elements).

Hence we will mostly work with the field of real numbers $\mathbb{F} = \mathbb{R}$, and there is no harm in viewing everything in this appendix in this context. However, you should be aware that everything here works over any field, $\mathbb{F}$. You should also be aware that if a linear system of equations over $\mathbb{R}$ has all of its coefficients and constants in $\mathbb{Q}$, the rational numbers (at times these coefficients and constants will all be integers), then the solutions to these systems can all be found by working in $\mathbb{Q}$; hence when we construct kernels, quotients, cokernels, etc., all the resulting matrices, vector spaces, and their bases, can be taken to have coeffients in $\mathbb{Q}$.[27]

D.3. **Vector Spaces.** Let $\mathbb{F}$ be a field. A *vector space over* $\mathbb{F}$ is a standard concept, that you can look up in many linear algebra textbooks (e.g., [J̈94, Axl15, HJ85, HJ13], all currently free for UBC library customers). One way to define a vector space is as follows: a *vector space over* $\mathbb{F}$ is triple $(V, +, \cdot)$ consisting of

(1) a set V (of *vectors*),
(2) a map $+\colon V \times V \to V$ (the *addition of vectors*, where $+(v_1, v_2)$ is usually denoted $v_1 + v_2$), and
(3) a map $\cdot\colon \mathbb{F} \times V \to V$ (the *scalar multiplication*, where $\cdot(\alpha, v)$ is usually denoted $\alpha \cdot v$ or just αv,

such that:

(1) The operation $+$ is commutative and associative[28], V has an identity element $0 = 0_V$ under $+$ (i.e., $0_V + v = v$ for all $v \in V$), and each $v \in V$ has an additive inverse $-v$ (i.e., $v + (-v) = 0_V$).[29],

[26]Briefly, a *field* is a triple $(\mathbb{F}, +, \cdot)$ where $+$ and $\cdot$ are binary operations (respectively, *addition* with $+(\alpha, \beta)$ denoted $\alpha + \beta$ and *multiplication* with $\cdot(\alpha, \beta)$ denoted $\alpha \cdot \beta$) on $\mathbb{F}$ that are commutative and associative (i.e., $\alpha + \beta = \beta + \alpha$ and $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ for all α, β, γ, and similarly with $+$ replaced with $\cdot$, such that $\mathbb{F}$ is a group under $+$ with identity element 0 (i.e., $0 + \alpha = \alpha$ for all $\alpha \in \mathbb{F}$, and any $\alpha \in \mathbb{F}$ has an additive inverse, denoted $-\alpha$, such that $\alpha + (-\alpha) = 0$), $\mathbb{F} \setminus \{0\}$ is a group under $\cdot$ with identity element 1 (where the multiplicative inverse of $\alpha \in \mathbb{F} \setminus \{0\}$ is denoted $1/\alpha$ or α^{-1}), and $\cdot$ distributes over $+$ in that $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$. Many properties follow from the above. Generally we also assume $1 \neq 0$, for otherwise we easily see that $\mathbb{F} = \{0\}$, which is of no interest to us or, generally, to linear algebra; every other field $\mathbb{F}$ has an interesting theory of linear algebra.

[27]More formally, if V is a vector space over the $\mathbb{Q}$ (see below), and if you know what is meant by the "tensor product" of two $\mathbb{F}$ vector spaces, then then $V \otimes_{\mathbb{Q}} \mathbb{R}$ is the "same" vector space where you view V as "living" over $\mathbb{R}$.

[28]See the previous footnote for *commutative* and *associative* and related terms.

[29]In other words, V is a commutative group under $+$.

(2) The operations $+$ and $\cdot$ satisfy the distributive laws

$$\alpha(v+v') = \alpha v + \alpha v', \quad (\alpha+\beta)v = \alpha v + \beta v$$

for all $\alpha, \beta \in \mathbb{F}$ and $v, v' \in V$.

We easily verify a number of other properties, e.g., $(0)v = 0_V$ and $(-1)v = -v$; we typically write 0 instead of 0_V if confusion is unlikely to occur.

If V, W are $\mathbb{F}$-vector spaces, a *linear transformation from V to W* is a map of sets $\mathcal{L}\colon V \to W$ such that for all $\alpha \in \mathbb{F}$ and $v, v' \in V$ we have

$$\mathcal{L}(\alpha v) = \alpha \mathcal{L}(v), \quad \mathcal{L}(v+v') = \mathcal{L}(v) + \mathcal{L}(v').$$

For $\mathbb{F}$-vector spaces V and W we have the following terminology and facts.

(1) We say that a subset $V' \subset V$ is a *subspace of V* if V' is closed under the $+$ and $\cdot$ operations of V, and therefore V' is a vector space under the restrictions of $+$ and $\cdot$ of V to V'.
(2) If $A, B \subset V$ are subsets, then we use the notation

$$A + B \stackrel{\mathrm{def}}{=} \{a + b \mid a \in A,\ b \in B\}.$$

(3) If $A \subset V$, the *span of A in V* is the subspace of V:

$$\mathrm{Span}_V(A) \stackrel{\mathrm{def}}{=} \{\alpha_1 v_1 + \cdots + \alpha_r v_r \mid \forall i \in [r],\ \alpha_i \in \mathbb{F}, v_i \in V\},$$

which can also be defined as (informally) the *smallest* subspace of V containing A, or (formally) the intersection of all subspaces of V containing A.
(4) If $V_1, V_2 \subset V$, then $V_1 + V_2$ is a subspace of V, equal to $\mathrm{Span}(V_1 \cup V_2)$.
(5) Let $U \subset V$ is be a subspace; the *quotient space* V/U refers to all sets of the form $v + U$ (also called the *U-coset of v*); the operations $+$ and $\cdot$ of V are well-defined on V/U, in that $(v_1 + U) + (v_2 + U) = (v_1 + v_2) + U$ (see the above definition of a $+$ of sets), and

$$\alpha \cdot (v_1 + U) \stackrel{\mathrm{def}}{=} \{\alpha(v_1 + u) \mid u \in U\}$$

equals $\alpha v_1 + U$; we understand V/U to be a vector space under these operations $+$ and $\cdot$ on U-cosets.
(6) If $\mathcal{L}\colon V \to W$ is a linear transformation, we define the *kernel* or *nullspace* of $\mathcal{L}$ to be

$$\ker(\mathcal{L}) \stackrel{\mathrm{def}}{=} \{v \in V \mid \mathcal{L}(v) = 0_W\} \subset V$$

which is a subspace of V, the *image* of W to be

$$\mathrm{Image}(\mathcal{L}) = \{\mathcal{L}(v) \mid v \in V\} \subset W$$

which is a subspace of W, and

$$\mathrm{coker}(\mathcal{L}) \stackrel{\mathrm{def}}{=} W/\mathrm{Image}(\mathcal{L})$$

which is a quotient space of W.

D.4. **Bases (the plural of basis), Dimension, Linear Systems.** The following ideas are used to define the dimension of a vector space.

If $\mathcal{B} \subset V$ is a subset, we say that:

(1) $\mathcal{B}$ is a *linearly independent set* if for any $b_1, \ldots, b_r \in \mathcal{B}$ and $\alpha_1, \ldots, \alpha_r \in \mathbb{F}$ we have
$$\alpha_1 b_1 + \cdots + \alpha_r b_r = 0$$
(hence $0 = 0_V$) implies that $\alpha_1 = \ldots = \alpha_r = 0$ (we alternatively say that *the elements of* $\mathcal{B}$ *are linearly independent*);
(2) $\mathcal{B}$ *spans* V if $\mathrm{Span}(\mathcal{B}) = V$;
(3) $\mathcal{B}$ is a *basis* of V if $\mathcal{B}$ is a linearly independent set and spans V.

The following properties are immediate when V is spanned by a finite set of vectors, which is equivalent to saying that V is *finite dimensional* (we define the dimension of V below). Otherwise, most of these properties require the Axiom of Choice, or, equivalently *transfinite induction*.[30]

(1) If $\mathcal{B}$ is a basis of V and $v \in V$ is non-zero, then there is a $b \in \mathcal{B}$ such that $\{v\} \cup (\mathcal{B} \setminus \{b\})$ is a basis of V.[31] Applying this repeatedly: if $\mathcal{A}$ is a subset of linearly independent vectors, then there is a subset $\mathcal{B}' \subset \mathcal{B}$ such that $|\mathcal{B}'| = |\mathcal{A}|$ (i.e., $\mathcal{B}', \mathcal{A}$ have the same cardinality or "size"), and $\mathcal{A} \cup (\mathcal{B} \setminus \mathcal{B}')$ is another basis of V; this is called the *basis exchange principle*).[32]
(2) If $S \subset V$ is a subset such that $\mathrm{Span}(S) = V$, then some subset of S is a basis for V.
(3) If $\mathcal{A}$ is a subset of linearly independent vectors, then there is a subset $\mathcal{A}'$ that is disjoint from $\mathcal{A}$ such that $\mathcal{A} \cup \mathcal{A}'$ is a basis for V.
(4) Any vector space, V, has a basis; the *dimension of* V, denoted $\dim(V) = \dim_{\mathbb{F}}(V)$ is cardinality of any basis of V, which is independent of the particular basis.

To see that any two bases of V have the same cardinality, you can use the basis exchange theorem. You can also prove this directly when $\dim(V) < \infty$, i.e., V is finite dimensional (or, equivalently, some finite subset of V spans all of V): to do

[30]When V is spanned by a some finite set of vectors, then the statements are true under the usual axioms of set theory. Otherwise, to prove that V has a basis, and that its cardinality does not depend on the particular basis, requires the Axiom of Choice (or, equivalently, transfinite induction). Similarly for the *basis exchange principle* when exchanging an infinite set of independent vectors into a basis. In homology theory, such as *singular homology*, we typically work with chains (and cochains) of vector spaces of (vastly) infinite dimension. On the other hand, we typically don't care about the Axiom of Choice and transfinite induction: we typically only write down bases for the homology groups, which are typically finite dimensional quotient subspaces or, at times, infinite dimensional but for which we can easily write down a basis. A good example of an uncountably infinite dimensional vector space is $\mathbb{R}$, viewed as a vector space over $\mathbb{Q}$ (more generally, if $\mathbb{F}'$ is a subfield of $\mathbb{F}$, then $\mathbb{F}$ is a vector space over $\mathbb{F}'$). A basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space is called a *Hamel basis*, and such a basis is typically used to do curious things, such as to construct a *non-measurable set*. In one of the exercises in these notes we use choose a basis for a subspace of $\mathbb{R}$ over $\mathbb{Q}$; however, fortunately this subspace is finitely generated, and hence has a finite basis. In fact, often when one uses transfinite induction (e.g., in proving the Hahn-Banach Theorem), in practice one only applies the induction finitely many (or countably infinitely many) times.

[31]This does not require the Axiom of Choice.

[32]If $\mathcal{A}$ is not countably infinite, then you generally need *transfinite induction* (which is equivalent to the Axiom of Choice) to prove this.

this, we can use some results regarding linear systems of equations. Here are the most useful results:

(1) Any $m \times m$ matrix, A, with entries in $\mathbb{F}$ can be brought into *row echelon form*, which allows one to solve the system $A\mathbf{x} = \mathbf{b}$ for the variable $\mathbf{x} \in \mathbb{F}^n$ and constant $\mathbf{b} \in \mathbb{F}^m$; alternatively we view $A\mathbf{x} = \mathbf{b}$ as a system of m equations in n unknowns, where $\mathbf{x}$ represents n *variables*, A are the *coefficients* of the equations, and $\mathbf{b} \in \mathbb{R}^m$ are the *constants* of the system.
(2) If $m < n$ are integers, then any system of m linear equations in n variables with coefficients in $\mathbb{F}$ that is homogeneous (i.e., the constants in the equations are all 0) has a nonzero solution.
(3) A system of n equations in n variables has a unique solution iff the corresponding homogeneous system has a unique solution, and this is equivalent to a number of conditions regarding the $n \times n$ matrix, M, of coefficients, such as: the *determinant* of M is nonzero; the rows of M are linearly independent as elements of $\mathbb{F}^n$; the same with "columns" replacing "rows;" M is an invertible matrix; the rank of M, meaning the dimension of the image of M, equals n; etc.

The fact that any two finite bases of a vector space have the same cardinality, and the basis exachange theorem, then follows from (2) above. However, some textbooks (e.g., [J9̈4]) go the other way: by proving the basis exchange theorem, one can directly prove (2) and (3); however, to solve systems of linear equations in practice, row echelon form (or some more limited form of row reduction) is useful in small examples.

D.5. **Coordinates.** Let V be a $\mathbb{F}$-vector space of dimension n (n finite), and $\mathcal{A} = \{v_1, \ldots, v_n\}$ be a basis, whose elements have therefore been arranged in some order. For $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ we write

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}_{\{v_1,\ldots,v_n\}} = \alpha_1 v_1 + \cdots + \alpha_n v_n,$$

which we view as giving "coordinates" for each element of V as a "column vector" in $\mathbb{F}^n$; we simply write $\mathcal{A}$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}_{\mathcal{A}} \quad \text{or} \quad \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

when $v_1, \ldots, v_n$ is understood, although we mention that the order of the elements $v_1, \ldots, v_n$ in $\mathcal{A}$ is crucial; we may call $\mathcal{A} = \{v_1, \ldots, v_n\}$ an *ordered basis* for emphasis.

Let $\mathcal{L}\colon V \to W$ be a linear map of V to an $m < \infty$ dimensional $\mathbb{F}$-vector space W with ordered basis $\mathcal{B} = \{w_1, \ldots, w_n\}$. Then we identify $\mathcal{L}$ with the $m \times n$ matrix

$$L = \begin{bmatrix} \ell_{11} & \cdots & \ell_{1n} \\ \vdots & \ddots & \vdots \\ \ell_{m1} & \cdots & \ell_{mn} \end{bmatrix}_{\mathcal{A},\mathcal{B}}$$

where the $\ell_{ij} \in \mathbb{F}$ are given (uniquely) by

$$\mathcal{L}(v_i) = \ell_{i1}w_1 + \cdots + \ell_{im}w_m,$$

so that in the usual rules of matrix multiplication we have

$$L \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}_{\mathcal{A}} = \left(\begin{bmatrix} \ell_{11} & \cdots & \ell_{1n} \\ \vdots & \ddots & \vdots \\ \ell_{m1} & \cdots & \ell_{mn} \end{bmatrix}_{\mathcal{A},\mathcal{B}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}_{\mathcal{A}} \right)_{\mathcal{B}}$$

In practice (and in these notes), to work with $\mathcal{L}$ in coordinates with respect to the bases $\mathcal{A}, \mathcal{B}$ it can be helpful to write the ℓ_{ij} in the table:

$$\begin{array}{c|ccc} & v_1 & \cdots & v_n \\ \hline w_1 & \ell_{11} & \cdots & \ell_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_m & \ell_{m1} & \cdots & \ell_{mn} \end{array} \quad \text{or} \quad \begin{array}{cccc} & v_1 & \cdots & v_n \\ w_1 & \ell_{11} & \cdots & \ell_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_m & \ell_{m1} & \cdots & \ell_{mn} \end{array}$$

D.6. **Some Additional Useful Concepts and Formulas.** In these notes, we will use the following concepts and formulas.

(1) If V is finite dimensional, then for any subspace $V' \subset V$ we have $\dim(V/V') = \dim(V) - \dim(V')$.

(2) For any linear transformation $\mathcal{L} \colon V \to W$ we define the *rank of* $\mathcal{L}$ to be $\mathrm{Rank}(\mathcal{L}) = \dim(\mathrm{Image}(\mathcal{L}))$, whose dimension is at most $\dim(V)$; $\mathcal{L}$ then gives an isomorphism $V/\ker(\mathcal{L}) \to \mathrm{Image}(\mathcal{L})$, and hence

$$\dim(V) = \dim(\ker(\mathcal{L})) + \dim(\mathrm{Image}(\mathcal{L})) = \dim(\ker(\mathcal{L})) + \mathrm{Rank}(\mathcal{L}). \tag{84}$$

(3) Let V, W be $\mathbb{F}$-vector spaces. We define *the direct sum of* V *and* (or *followed by*) W, to be

$$V \oplus W = V \times W = \{(v, w) \mid v \in V,\ w \in W\}$$

which becomes a vector space by applying the $+$ and $\cdot$ operations "component by component." More generally one can define the direct sum of any finite number of vector spaces.[33] We have

$$\dim(V \oplus W) = \dim(V) + \dim(W)$$

(for finite dimensional space, V, W, but which also holds, appropriately interpreted, for arbitrary vector spaces).

(4) There is also a very important *tensor product*, $V \otimes W$ (for which $\dim(V \otimes W) = \dim(V)\dim(W)$, but we will not need this here.

[33] We warn the reader that if we have an infinite number of vector spaces $\{V_i\}_{i \in I}$, then their *product* refers to vector space based on their cartesian product, i.e., $\prod_{i \in I} V_i$, while their direct sum refers to the subspace of their product

$$\bigoplus_{i \in I} V_i \stackrel{\text{def}}{=} \left\{ (v_i)_{i \in I} \in \prod_{i \in I} V_i \;\middle|\; v_i = 0 \text{ for all but finitely many value of } i \in I \right\}.$$

(5) If $V_1, \ldots, V_r \subset V$ are subspaces, we say that V *equals the internal sum* $V_1 \oplus \cdots \oplus V_r$ if any vector in V can be uniquely written as a sum of vectors in $V_1, \ldots, V_r$; equivalently, the map

$$f \colon V_1 \oplus \cdots \oplus V_r \to V \quad \text{given by } f(v_1, \ldots, v_r) = v_1 + \cdots + v_r$$

(where the domain of f is the usual direct sum) is an isomorphism. In this case we have

$$\dim(V) = \dim(V_1) + \cdots + \dim(V_r);$$

moreover, if $\mathcal{A}_1, \ldots, \mathcal{A}_r$ are bases for, respectively, $V_1, \ldots, V_r$, then the $\mathcal{A}_i$ are disjoint and $\mathcal{A} = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_r$ is a basis for V. If W is another vector space, and W equals the internal direct sum $W_1 \oplus \cdots \oplus W_s$, then choosing bases $\mathcal{B}_1, \ldots, \mathcal{B}_s$ for the $W_1, \ldots, W_s$, the matrix repsentation of any linear transformation $\mathcal{L} \colon V \to W$ naturally decomposes into an $r \times s$ *block matrix*, whose i, j-th block is a $|\mathcal{B}_i| \times |\mathcal{A}_j|$ matrix. Block matrices are sometimes called *partitioned matrices*; see Section 0.7 of *Matrix Analysis* by Horn and Johnson ([HJ13, HJ90, HJ85]).

D.7. **Formal $\mathbb{R}$-Linear Combinations of a Set.** The definition we gave of $\mathbb{R}[S]$ for a set, S, of *formal $\mathbb{R}$-linear sums of S* (Definition 4.1) is not entirely precise. Here is one very precise (but tedious) definition of these formal sums.

Definition D.1 (Tedious But Precise Definition of Formal Sums)**.** Let S be any set. An $\mathbb{R} \times S$ *sequence* is a (possibly empty) sequence of elements of $\mathbb{R} \times S$, i.e., a sequence $\bigl((\alpha_1, s_1), \cdots, (\alpha_r, s_r)\bigr)$, which we write in shorthand as

$$\alpha_1 s_1 \boldsymbol{+} \cdots \boldsymbol{+} \alpha_r s_r \tag{85}$$

(hence the $\boldsymbol{+}$ is just notational shorthand, at least temporarily). We say two such sequences are equivalent, writing

$$\alpha_1 s_1 \boldsymbol{+} \cdots \boldsymbol{+} \alpha_r s_r \sim \alpha_1' s_1' \boldsymbol{+} \cdots \boldsymbol{+} \alpha_{r'}' s_{r'}',$$

if for each $s \in S$, the sum of the α_i over those i with $s_i = v$ equals the sum of the $\alpha_{i'}'$ over those i' with $s_{i'}' = s$; it is immediate that $\sim$ is an equivalence relation. A *formal $\mathbb{R}$-linear sum in S* refers to an equivalence class of $\mathbb{R} \times S$-sequences; we use $\mathbb{R}[S]$ denote the set of all formal $\mathbb{R}$-linear sums in S We define the sum $(+)$ on $\mathbb{R} \times S$ sequences as the concatenation of squences,

$$\bigl(\alpha_1 s_1 \boldsymbol{+} \cdots \boldsymbol{+} \alpha_r s_r\bigr) + \bigl(\alpha_1' s_1' \boldsymbol{+} \cdots \boldsymbol{+} \alpha_{r'}' s_{r'}')\bigr) \stackrel{\text{def}}{=} \alpha_1 s_1 \boldsymbol{+} \cdots \boldsymbol{+} \alpha_r s_r \boldsymbol{+} \alpha_1' s_1' \boldsymbol{+} \cdots \boldsymbol{+} \alpha_{r'}' s_{r'}',$$

and we see that $+$ is well-defined on equivalence classes of $\mathbb{R}[S]$, and is associative and commutative. By definition of $+$, the equivalence class of (85) equals

$$\alpha_1 s_1 + \cdots + \alpha_r s_r, \tag{86}$$

and this is the usual way we write elements of $\mathbb{R}[S]$. (Hence a posteriori, the $\boldsymbol{+}$ becomes $+$ in $\mathbb{R}[S]$.) Defining a scalar multiplication

$$\beta\bigl(\alpha_1 s_1 + \cdots + \alpha_r s_r\bigr) \stackrel{\text{def}}{=} \beta\alpha_1 s_1 + \cdots + \beta\alpha_r s_r$$

turns $\mathbb{R}[S]$ into an $\mathbb{R}$-vector space.

D.8. **Examples of Maps of Quotient Spaces.** Many students in CPSC 531F have seen quotient spaces abstractly, but most will never have needed to use them. Here we give the usual intuition in working with quotient spaces, review their definition (see Subsection 4.7.3 and the definition earlier in this appendix), and provide examples to show that a linear map $\mathcal{L}\colon U \to W$ of $\mathbb{R}$-vector spaces extends to a map of quotient spaces $U/U_1 \to W/W_1$ when (and only when) $\mathcal{L}(U_1) \subset W_1$.

We will need these remarks in Subsection 5.7, when we need this in the Mayer-Vietoris sequence.

(If $U_1 \subset U$ are $\mathbb{R}$-vector spaces, one can identify U/U_1 with any $U_2 \subset U$ such that $U_1 \cap U_2 = 0$ and $U_1 + U_2 = U$. An important example of such a U_2 is $U_1^\perp$, the orthogonal complement of U_1. However, it is truer to say that the elements of U_2 are "coset representatives" of U/U_1, in that each element of U/U_1 is uniquely of the form $u_2 + U_1$, where $u_2 \in U_2$. There is often no harm in identifying U/U_1 with U_2. However, when we want to describe maps of quotient spaces, such as what we will need in the Mayer-Vietoris sequence, it becomes much simpler to work with the spaces U/U_1. The reason is that the maps in the Mayer-Vietoris sequence are canonical when working with quotient spaces, but become very awkward if we choose a set of coset representatives of U/U_1.)

It is also important to understand that while textbooks often define U/U_1 as U_1-coset spaces, i.e., subsets of the form $u + U_1$, we just as often think of U/U_1 as representing *equivalence classes* of the *equivalence relation* $u \sim u'$ iff $u - u' \in U_1$. Here we will provide some examples.

D.8.1. *"Real World" Intuition, or "Math Stories".* It is important to conceptualize mathematics — when possible — in terms of "real world" examples. So, for example, when you write down the formula $(AB)^{-1} = B^{-1}A^{-1}$ for invertible, compatible matrices, you can prove the formula, or give a 2×2 example to show that $(AB)^{-1}$ is not generally equal to $A^{-1}B^{-1}$. However, the typical "real world" example of this is that in the morning, many of us put our socks on first, and then our shoes, but to reverse this process we first take off our shoes, not our socks.

Here we will do our best with equivalence relations and coset spaces.

Example D.2. If you are currently in Vancouver or Burnaby (in Canada), you are also in British Columbia; however, if you are in British Columbia, you can't tell whether you are in Vancouver or some other city. Similarly, the class of an integer modulo 6 determines its class modulo 3, but not vice versa. In computer science one often thinks of "the integers modulo 3" as a set $\{0, 1, 2\}$, and of "mod 3" as a map $\mathbb{Z} \to \{0, 1, 2\}$ (taking n to $\mathrm{mod}(n, 3)$). From this point of view, knowing an integer mod 6 tells us its value mod 3, an represents a "refinement of information;" of course, knowing an integer's value mod 3 does not tell you its value mod 6 (but if $\mathrm{mod}(n, 3) = 2$, then you know that $\mathrm{mod}(n, 6)$ is either 2 or 5).

Example D.3. If you are currently in Vancouver or Burnaby (in Canada), you don't know if you are currently drinking coffee or not, and vice versa. Similarly, knowing $\mathrm{mod}(n, 3)$ does know tell you anything about $\mathrm{mod}(n, 4)$ (in the absence of other information).

Example D.4. Say that you celebrated your birthday one day late. If your birthday is January 1, then this celebration occurred on January 2. However, if your birthday is February 28, then this celebration occurred on either February 29 or March 1. Also, if your birthday is January 1 (or February 28), you can't really tell

(1) on which day of the week you were born, or (2) whether or not you are currently drinking coffee. By constrast, if you know your birthday is January 1, then you do know the month during which you were born.

D.8.2. *Equivalence Relations, Equivalence Classes, and Quotients.* If S is a set, a *relation* is an arbitrary subset $R \subset S \times S$; R is called an *equivalence relation* if it is is reflexive (i.e., $(s,s) \in R$ for all $s \in S$), symmetric (i.e., $(s,s') \in R$ implies $(s',s) \in R$), and transitive (i.e., $(s,s'),(s',s'') \in R$ implies that $(s,s'') \in R$; also, we typically write $s \sim_R s'$ to mean that $(s,s') \in R$, or just $s \sim s'$ if R is understood. An *equivalence class of R (in S)* refers to any set of the form

$$[s]_{\sim_R} \stackrel{\text{def}}{=} \{s' \mid s' \sim_R s\},$$

and each element of S is in a unique equivalence class. We use $S/\sim_R$ (or simply $S/\sim$ if R is understood) to denote the set of equivalence classes.

Example D.5. Let P be a set of people. For $p,p' \in P$, say that we take $p \sim p'$ to mean that p,p' have the same (Gregorian calendar) birthday (more formally, we take $R \subset P \times P$ to be the set of pairs (p,p') such that p and p' have the same birthday). In this way P is partitioned into 366 equivalence classes, $P/\sim$. Knowing in which equivalence class p lies tells you p's birthday, but does not (generally) tell you if p is currently drinking coffee or not (without additional knowledge).

Example D.6. Say that for $n,n' \in \mathbb{Z}$ we write $n \sim_3 n'$ to mean that $n-n'$ is divisible by three, or equivalently

$$n - n' \in 3\mathbb{Z} = \{\ldots,-6,-3,0,3,6,\ldots\}.$$

Then $\sim_3$ is an equivalence relation; we commonly use

$$\mathbb{Z}/3\mathbb{Z} \quad \text{to denote} \quad \mathbb{Z}/\sim_3,$$

whose elements are

$$\begin{aligned} 0+3\mathbb{Z} &= \{\ldots,-6,-3,0,3,6,\ldots\}, \\ 1+3\mathbb{Z} &= \{\ldots,-5,-2,1,4,7,\ldots\}, \\ 2+3\mathbb{Z} &= \{\ldots,-4,-1,2,5,8,\ldots\}, \end{aligned}$$

and we use the notation

$$n+3\mathbb{Z} \stackrel{\text{def}}{=} \{n+m \mid m \in 3\mathbb{Z}\}.$$

So the entire set $\mathbb{Z}$ is partitioned into these three sets. The reason that the equivalence classes can be $\mathbb{Z}/3\mathbb{Z}$ makes sense is that $\mathbb{Z}$ is a group under $+$, and $3\mathbb{Z}$ is a subgroup; in this case the equivalence classes "modulo 3" will always be of the form $n+3\mathbb{Z}$ for some n; note also that the following are all equivalent:

$$n \sim_3 n',\ [n]_{\sim_3} = [n']_{\sim_3},\ n-n' \in 3\mathbb{Z},\ n+3\mathbb{Z} = n'+3\mathbb{Z}.$$

and

$$n \equiv n' \pmod 3,$$

and the reason that $n-n' \in 3\mathbb{Z}$ is equivalent to $n+3\mathbb{Z} = n'+3\mathbb{Z}$ is that $3\mathbb{Z}+3\mathbb{Z} = 3\mathbb{Z}$.

Example D.7. For the same reason, if $U_1 \subset U$ is a subspace, then relation $u \sim_{U_1} u'$ (or $u \sim u'$ when U_1 is understood) given by $u \sim u'$ iff $u - u' \in U_1$ is an equivalence relation, and the following are all synonymous:

$$u \sim_{U_1} u',\ [u]_{\sim_{U_1}} = [u']_{\sim_{U_1}},\ u - u' \in U_1,\ u + U_1 = u' + U_1.$$

and

$$u \equiv u' \pmod{U_1},$$

Then the usual definition of U/U_1 (see Subsection 4.7.3) is the set of U_1-cosets, i.e., sets of the form $[u]_{U_1} = u + U_1$. For example, let $U_1 \subset U = \mathbb{R}^2$ be the subspace

$$U_1 = \{(x, x) \mid x \in \mathbb{R}\}.$$

Then the elements of U/U_1 are the set of U_1-cosets, which includes

$$U_1 = (0,0) + U_1 = (-2,-2) + U_1,\ (1,0) + U_1 = (2,1) + U_1,\ (0,3) + U_1,\ (0,4) + U_1,$$

each of which is a line in $\mathbb{R}^2$ with slope 1 (DRAW PICTURE). We equivalently write $(x,y) \sim (x',y')$ if $(x'-x, y'-y) \in U_1$, which is equivalent to $x' - x = y' - y$.

D.8.3. *Morphisms of Quotient Spaces.* Now we claim that if $\mathcal{L} \colon U \to W$ is a linear map, then there is a natural way to "extend" $\mathcal{L}$ to quotient spaces $U/U_1 \to W/W_1$ (where U_1, W_1 are subspaces of U, W) iff $\mathcal{L}(U_1) \subset W_1$.

Example D.8. Let's work again in modular arithmetic. So let $U = W = \mathbb{Z}$, and $\mathcal{L} = \mathrm{id}_{\mathbb{Z}}$ be the identity map. Since

$$\mathcal{L}(6\mathbb{Z}) = 6\mathbb{Z} \subset 3\mathbb{Z},$$

there is a map $\mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$: to determine this map on $5 + 6\mathbb{Z} \in \mathbb{Z}/6\mathbb{Z}$, we see that

$$u \in 5 + 6\mathbb{Z} \quad \Rightarrow \quad \mathcal{L}(u) \in 2 + 3\mathbb{Z};$$

another way to understand this is that

$$\forall u, u' \in 5 + 6\mathbb{Z}, \quad \mathcal{L}(u) - \mathcal{L}(u') \in 3\mathbb{Z},$$

which implies that $\mathcal{L}(u), \mathcal{L}(u')$ are in the same equivalence class in $\mathbb{Z}/3\mathbb{Z}$.

Let us give some more examples along these lines.

Example D.9. Say that $U = W = \mathbb{Z}$, and $\mathcal{L} \colon U \to W$ is the identity map $\mathcal{L} = \mathrm{id}_{\mathbb{Z}}$ (i.e., $\mathcal{L}(u) = u$). If $U_1 = 6\mathbb{Z}$ and $W_1 = \{0\}$, note that $\mathcal{L}(U_1)$ is not contained in W_1 (since $\mathcal{L}(U_1) = 6\mathbb{Z}$ is not a subset of $W_1 = \{0\}$); note also that $\mathcal{L}$ doesn't extend to map

$$U/U_1 = \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z} = W/W_1,$$

since, for example, the element $1 + 6\mathbb{Z}$ of U/U_1 has

$$\mathcal{L}(1 + 6\mathbb{Z}) = \mathcal{L}\bigl(\{\ldots, -5, 1, 7, 13, \ldots\}\bigr) = \{\ldots, -5, 1, 7, 13, \ldots\},$$

which is not a single element of $W/W_1 = \mathbb{Z}$. However, if we do the same but instead take $W_1 = 3\mathbb{Z}$, then $\mathcal{L}$ does extend to a map

$$U/U_1 = \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} = W/W_1,$$

since $\mathcal{L}(U_1) = 6\mathbb{Z} \subset 3\mathbb{Z} = W_1$, or, by example

$$\mathcal{L}(1{+}6\mathbb{Z}) = \mathcal{L}\bigl(\{\ldots, -5, 1, 7, 13, \ldots\}\bigr) = \{\ldots, -5, 1, 7, 13, \ldots\} \subset 1{+}3\mathbb{Z} = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\}.$$

Similarly:

(1) the identity map $\mathcal{L} \colon \mathbb{Z} \to \mathbb{Z}$ does not extend to a map $\mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$; but

(2) the map $\mathcal{L}\colon \mathbb{Z} \to \mathbb{Z}$ given by $\mathcal{L}(u) = 2u$ does extend to a map $\mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$, since $\mathcal{L}(3\mathbb{Z}) \subset 6\mathbb{Z}$, or, by example
$$\mathcal{L}(1 + 3\mathbb{Z}) = 2 + 6\mathbb{Z} \subset \mathbb{Z}/6\mathbb{Z}.$$

Example D.10. We can model the previous example in the setting of vector spaces.

(1) The identity map $\mathcal{L}\colon \mathbb{F}^2 \to \mathbb{F}^2$ does not extend to a map $\mathbb{F}^2/U_1 \to \mathbb{F}^2/W_1$ where $U_1 = \{(x,x) | x \in \mathbb{R}\}$ and $W_1 = \{0\}$, since $\mathcal{L}(U_1)$ is not a subset of W_1; by example,
$$(0,3) + U_1 = \{(x, 3+x) | x \in \mathbb{R}\},$$
and $\mathcal{L}$ maps this set to itself, which is not an element of $\mathbb{F}^2/W_1 \simeq \mathbb{F}^2$.

(2) However, the linear map $\mathcal{L}(x,y) = (2x-2y, y-x)$ does take U_1 to $\mathcal{L}(U_1) = W_1 = \{0\}$, and hence gives a map $\mathbb{F}^2/U_1 \to \mathbb{F}^2/W_1$; by example,
$$\mathcal{L}\big((0,3) + U_1\big) = (-6,3) + \mathcal{L}(U_1) = (-6,3) + \{0\} = \{(-6,3)\} = (-6,3) + W_1.$$

Example D.11. Give a similar example (or put one in the exercises) with $U = W = \mathbb{R}^2$,
$$U_1 = \{(x,x) | x \in \mathbb{R}\}, \quad W_1 = \{(x,-2x) | x \in \mathbb{R}\}.$$
A map $\mathcal{L}\colon \mathbb{R}^2 \to \mathbb{R}$ does not generally give a map $\mathbb{R}^2/U_1 \to \mathbb{R}^2/W_1$, but a map with $\mathcal{L}(U_1) \subset W_1$, such as the map $\mathcal{L}(x,y) = (x+y, -5x+y)$.

MORE MATERIAL MAY BE ADDED HERE, IF NEEDED.

D.9. **Exercises.**

Exercise D.1. Let $C^\infty(\mathbb{R})$ denote the set of functions $f\colon \mathbb{R} \to \mathbb{R}$ that are infinitely differentiable.

D.1(a) Explain briefly why the usual meaning of "addition of functions" and "multiplying a function by a real number" turn $C^\infty(\mathbb{R})$ into a $\mathbb{R}$-vector space. (Verify a few of the vector space axioms without writing and verifying every single axiom — the choice is yours.)

D.1(b) Let Trans_1 be the map from $C^\infty(\mathbb{R})$ to itself taking $f = f(x)$ to the function $g = \mathrm{Trans}_1(f)$ given by $g(x) = f(x+1)$. Show that Trans_1 (*translation by* $+1$) is a linear map.

D.1(c) Let D be the map from $C^\infty(\mathbb{R})$ to itself taking $f = f(x)$ to $(Df)(x) = f'(x)$ (i.e., the derivative) is a linear map.

D.1(d) Let M_x be the map from $C^\infty(\mathbb{R})$ to itself taking $f = f(x)$ to $(M_x f)(x) = xf(x)$ (i.e., multiplication by x) is a linear map.

D.1(e) A *linear operator* on an $\mathbb{F}$-vector space, V, is any linear transformation $V \to V$. If A, B are any two linear operators on V, show that $AB = A \circ B$ is another linear transformation on V, and that so is $[A,B] \stackrel{\text{def}}{=} AB - BA$.

D.1(f) What is the operator on $C^\infty(\mathbb{R})$ given by $[\mathrm{Trans}, D]$ equals the zero tranformation (taking f to 0, the zero function). Explain this in intuitive terms.

D.1(g) Show that the operator on $C^\infty(\mathbb{R})$ given by $[D, M_x]$ equals the *identity operator*, i.e., id, taking f to f. This is one form of the *Heisenberg uncertainty principle*.

Exercise D.2. For each $i \in \{0, 1, \ldots\}$, we speak of $f(x) = x^i$ as the function taking x to x^i; hence this is an element of $C^\infty(\mathbb{R})$ as above. Say that $d \in \{0, 1, \ldots\}$. A function $f : \mathbb{R} \to \mathbb{R}$ is a called *polynomial of degree at most* d if for some $\alpha_0, \ldots, \alpha_d \in \mathbb{R}$ we have

$$f(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_d x^d$$

as functions; we use P_d to denote the set of all polynomials of degree d; we easily see that P_d is a subspace of $C^\infty(\mathbb{R})$. Rolle's Theorem implies that if $p(x)$ is a polynomial of degree d, and $p(x) = 0$ for $d + 1$ distinct values of $x \in \mathbb{R}$, then $p = 0$, i.e., p is the zero polynomial. It is also a standard fact that if $f \in C^\infty(\mathbb{R})$ has $f^{(d+1)}$, the $(d + 1)$-st derivative of f, equals the 0 function, then $f \in P_d$.

D.2(a) Show that the functions $1, x, \ldots, x^d$ are linearly independent in $C^\infty(\mathbb{R})$, i.e., if $\alpha_0, \ldots, \alpha_d \in \mathbb{R}$ and

$$f(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_d x^d$$

equals 0 (the zero function), then $\alpha_0 = \ldots = \alpha_d = 0$.

D.2(b) Conclude that $\dim(P_d) = d + 1$.

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