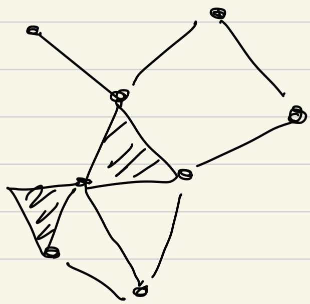
CPSC 531F Jan 8, 2025

- Last time:

  - Simplex in $\mathbb{R}^N$

  - Simplicial Complex, K, in $\mathbb{R}^N$



in $\mathbb{R}^2$

Rem: Definition I gave in class last time was incorrect...

- Today: (1) Correct the above

  (2) Graphs and Simplicial Complexes

Admin:

Notes posted to course website

https://www.cs.ubc.ca/~jf

etc.

I usually collect homework at

- the end of the course
- sometime in the middle
- OR see me if you have questions

- Remark: Prof. Klaus Hoechsmann used to teach linear algebra as follows:

(1) 1^{st} 2 weeks: the entire course, restricted to 2×2 matrices and 2×2 systems, and

(2) Weeks 3 and on: the entire course again, but in the general case.

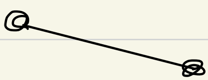
We will often take a similar approach ...

Recall:

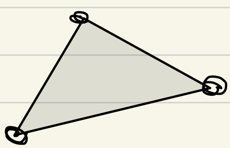
$$d\text{-simplex} = \mathrm{conv}(\underbrace{\vec{s}_0, \ldots, \vec{s}_d}_{\substack{d+1 \text{ vectors} \\ \text{in general position}}})$$

in $\mathbb{R}^N$

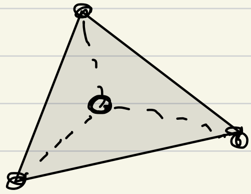
0-simplex



1-simplex



2-simplex



3-simplex

Formally: we define:

general position CPSC 531F

(and Armstrong: Basic Topology (free))

(**geometrically independent** in Munkres: Elements of Algebraic Topology (not free, excellent))

(**affinely independent** in Matousek: Using the Borsuk-Ulam Theorem)

We defined:

$$\operatorname{conv}(\vec{s}_0, \dots, \vec{s}_d)$$

$$= \left\{ \alpha_0 \vec{s}_0 + \alpha_1 \vec{s}_1 + \dots + \alpha_d \vec{s}_d \;\middle|\; \begin{array}{l} \alpha_0, \dots, \alpha_d \in \mathbb{R}, \quad \alpha_i \geq 0 \\ \alpha_0 + \dots + \alpha_d = 1 \end{array} \right\}$$

$(\alpha_0, \dots, \alpha_d)$ also called <u>stochastic</u>

General position: $\vec{s}_0, \dots, \vec{s}_d$ in general position if

$\vec{s}_1 - \vec{s}_0, \dots, \vec{s}_d - \vec{s}_0$ are linearly independent

<u>Exercise</u>: Show that if

$$\vec{x}_1 = (1,1,1) \in \mathbb{R}^3$$

$$\vec{x}_2 = (2,4,8) \in \mathbb{R}^3$$

$$\vdots$$

$$\vec{x}_n = (n, n^2, n^3) \in \mathbb{R}^3$$

$$\vdots$$

then any four of these vectors are in general position.

Can you produce vectors in $\mathbb{R}^2$, $\vec{y}_1, \vec{y}_2, \ldots$ s.t. any 3 are in general position?

Hint: We know that a

4×4 Vandermonde matrix

$$\begin{bmatrix} 1 & a_1 & a_1^2 & a_1^3 \\ 1 & a_2 & a_2^2 & a_2^3 \\ 1 & a_3 & a_3^2 & a_3^3 \\ 1 & a_4 & a_4^2 & a_4^3 \end{bmatrix}$$

is invertible, $\det \neq 0$ if

$a_1, a_2, a_3, a_4 \in \mathbb{R}$

are distinct

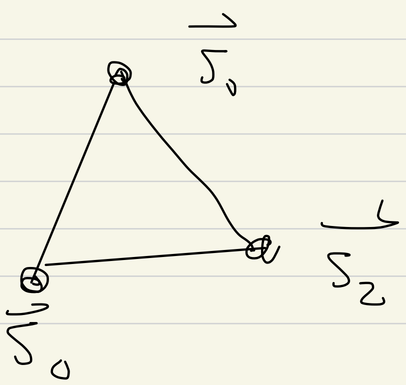
Simplex d-dimensional

$\mathrm{Conv}(\vec{s}_0, \dots, \vec{s}_d)$ with

$\vec{s}_0, \dots, \vec{s}_d$ in general position.

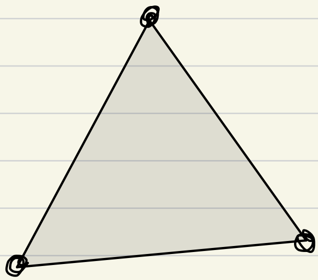
$\mathrm{Conv}(\vec{s}_0, \dots, \vec{s}_d)$

$$\left\{ \alpha_0 \vec{s}_0 + \dots + \alpha_d \vec{s}_d \;\middle|\; \begin{array}{c}(\alpha_0, \dots, \alpha_d) \\ \text{is} \\ \text{stochastic}\end{array} \right\}$$



$\subset \mathbb{R}^N$

Claim: Given



$\subset \mathbb{R}^N$

this subset uniquely determines

$\{ \vec{s}_0, \vec{s}_1, \vec{s}_2 \}$ s.t.

$\mathrm{Conv}(\vec{s}_0, \vec{s}_1, \vec{s}_2)$

Exercise: Show that if

$\vec{s}_0, \vec{s}_1, \vec{s}_2, \vec{t}_0, \vec{t}_1, \vec{t}_2$ s.t.

$\mathrm{Conv}(\vec{s}_0, \vec{s}_1, \vec{s}_2) = \mathrm{Conv}(\vec{t}_0, \vec{t}_1, \vec{t}_2)$

and $\vec{s}_0, \vec{s}_1, \vec{s}_2$ in general position

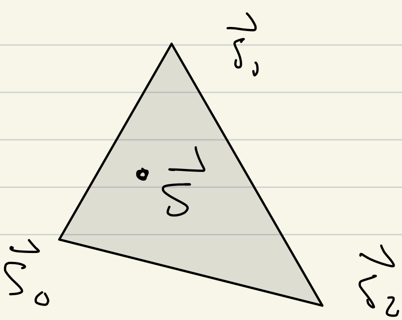
then $\vec{t}_0, \vec{t}_1, \vec{t}_2$ are " " "

and

$$\{\vec{s}_0, \vec{s}_1, \vec{s}_2\} = \{\vec{t}_0, \vec{t}_1, \vec{t}_2\}.$$

Hint!

If $\vec{s} \in \mathrm{Conv}(\vec{s}_0, \vec{s}_1, \vec{s}_2)$



by definition

$$\vec{s} = \alpha_0 \vec{s}_0 + \alpha_1 \vec{s}_1 + \alpha_2 \vec{s}_2$$

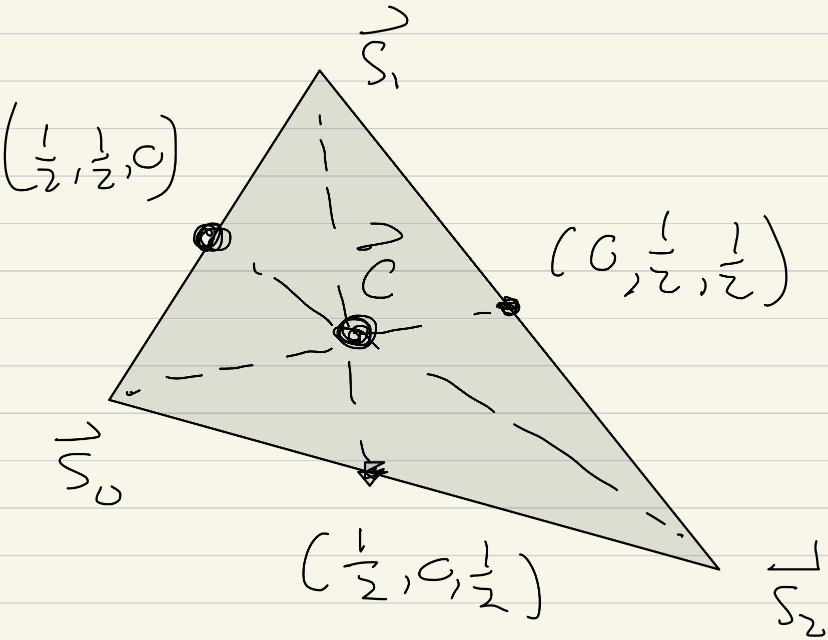
s.t. $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}_{\geq 0}$

$$\alpha_0 + \alpha_1 + \alpha_2 = 1.$$

Exercise: Show that the triple $(\alpha_0, \alpha_1, \alpha_2)$ is unique (assuming $\vec{s}_0, \vec{s}_1, \vec{s}_2$ are in general position).

This triple $(\alpha_0, \alpha_1, \alpha_2)$ is called the <u>Barycentric</u> <u>coordinates</u>

of $\vec{s}$ with respect to $\vec{s}_0, \vec{s}_1, \vec{s}_2$



$\vec{c}$ = centre of mass

$= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ in Barycentric coordinates

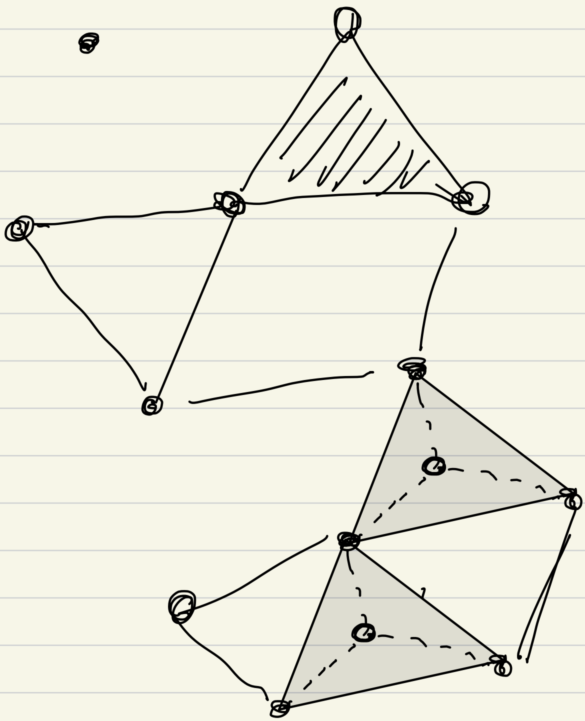
Simplicial complex:

K a set whose elements are simplices in $\mathbb{R}^N$ s.t.

(1) if $S = \mathrm{Conv}(\vec{\alpha}_0, \dots, \vec{\alpha}_d) \in K$ then so is the convex hull of any subset

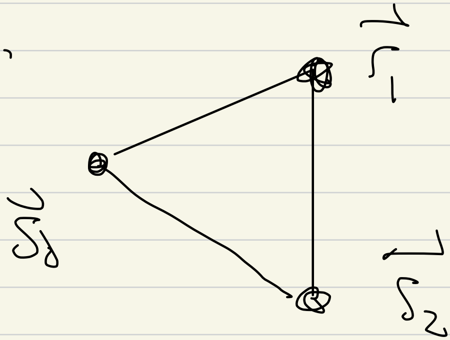
(2) $S, S' \in K$, then $S \cap S'$ is a "face" of S, S'

where a "face" of $\mathrm{Conv}(\vec{\alpha}_0, \dots, \vec{\alpha}_d)$ is Conv(any subset of $\vec{\alpha}_0, \dots, \vec{\alpha}_d$)



$\mathbb{R}^N$

E.g.,



$\subset \mathbb{R}^N$

$$K = \{ \emptyset, \{\vec{s}_0\}, \{\vec{s}_1\}, \{\vec{s}_2\},$$

$$\mathrm{Conv}(\vec{s}_0, \vec{s}_1), \mathrm{Conv}(\vec{s}_1, \vec{s}_2),$$

$$\mathrm{Conv}(\vec{s}_0, \vec{s}_2) \}$$

Looks like a graph!

$G = (V, E)$, $V = \{\vec{s}_0, \vec{s}_1, \vec{s}_2\}$

E = all sets of size 2 of V

So! Abstract simplicial complex:

$(\bar{V}, K_{abs})$

$\bar{V}$ is a finite set

K_{abs} a set of subsets of $\bar{V}$

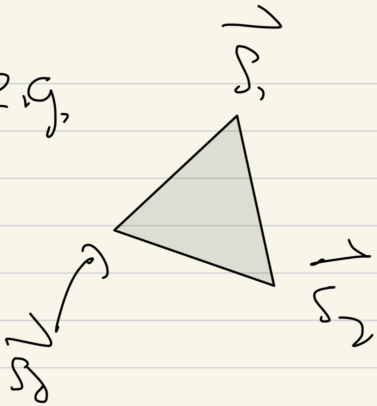
s.t. if $A \in K_{abs}$

and $A' \subseteq A$, then $A' \in K_{abs}$

K_{abs} contains $\{v\}$ for all $v \in \bar{V}$

$\bar{V}$ "vertex set of K_{abs}."

e.g.,



$\subset \mathbb{R}^4$

$$S = \mathrm{Conv}(\vec{s}_0, \vec{s}_1, \vec{s}_2)$$

The largest simplicial complex K, with vertex set $V = \{\vec{s}_0, \vec{s}_1, \vec{s}_2\}$ is

$$K = \Big\{ \emptyset, \{\vec{s}_0\}, \{\vec{s}_1\}, \dots, \mathrm{conv}(\vec{s}_0, \vec{s}_1), \dots, \mathrm{conv}(\vec{s}_0, \vec{s}_1, \vec{s}_2) \Big\}$$

$$K_{abs} = \{ \emptyset, \{\vec{s}_0\}, \{\vec{s}_1\}, \{\vec{s}_0, \vec{s}_1\}, \ldots, \{\vec{s}_0, \vec{s}_1, \vec{s}_2\} \}$$

$$K \leftrightarrow K_{abs}$$

combinatorial thing

a collection of simplices

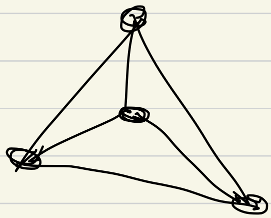
known $\vec{s}_0, \vec{s}_1, \vec{s}_2 \in \mathbb{R}^N$

If K_{abs} is abstract simp complex, we say that the $\dim(K_{abs}) = d$ if the largest size of a set in K_{abs} is size $d+1$.

Graph (simple graph)

$\longleftrightarrow$ 1-dim abst simplicial complex

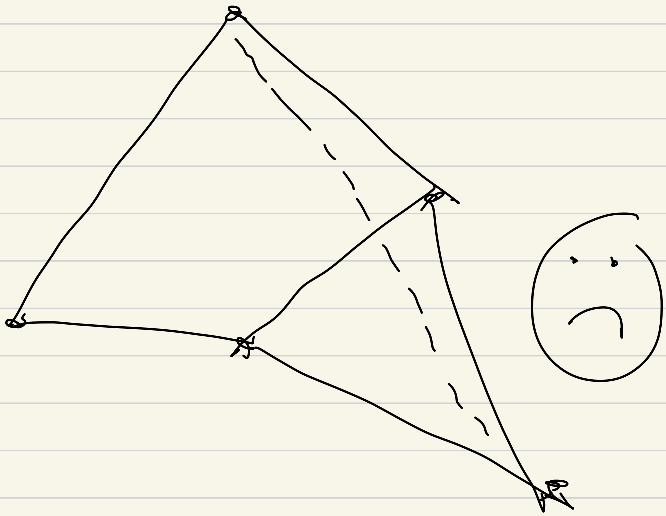
Example



is a simplicial complex, K, in $\mathbb{R}^2$

Claim: If K is a simplicial complex in $\mathbb{R}^2$, K_{abs} is the abstract complex, and is dimension 1, then

$K_{abs} \neq$ complete graph on 5 vertices



$\mathbb{R}^2$

<u>Thm</u>: Any graph $G=(V,E)$ is the abst complex of a complex $K \subset \mathbb{R}^2$?