CPSC 531F Jan 10, 2025

Today:

- Any simple graph, $G=(V,E)$ is the abstract simplicial complex of a simplicial complex in $\mathbb{R}^3$

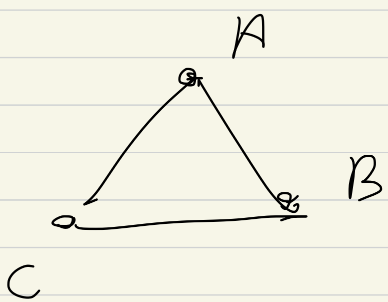
- Define simplicial homology; start with graphs

Given a <u>simple</u> graph

$G = (V, E)$:

V is a set (finite, unless otherwise indicated)

E is a collection of subsets of V of size 2 :

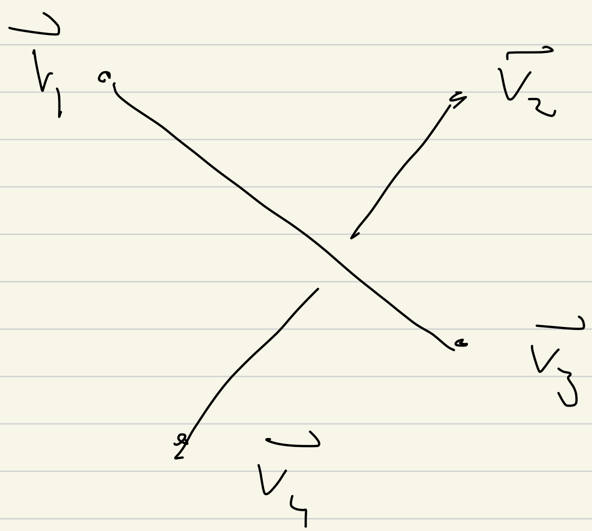


$V = \{A, B, C\}$

$$E = \left\{ \{A, B\}, \{B, C\}, \{A, C\} \right\}$$

Idea: $\vec{V} \to \mathbb{R}^3$

$$\{v_1, \ldots, v_n\}$$
$$\downarrow$$
$$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^3$$



E some collection of sets size 2

Simplicial complex in $\mathbb{R}^3$:

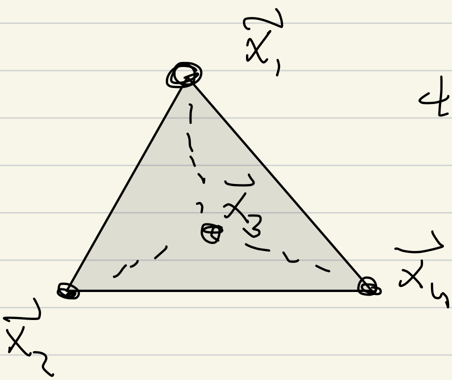
$K = \{ \text{simplices in } \mathbb{R}^3 \}$

s.t.

(1) $S \in K$, S' is a face of S

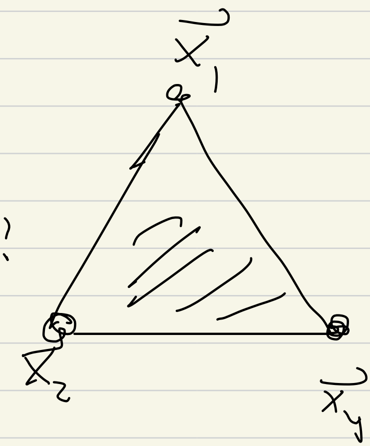
then $S' \in K$

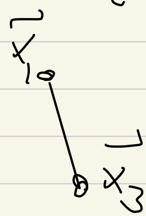
S



then

add:





add

(2) "Crafty"

$S, S' \in K$, then

$S \cap S'$ is a face of S, and of S'.

=

To check (2):

V	E
$v_1 \longmapsto \vec{x}_1$	$\{v_1, v_2\} \in E$ then
$v_2 \longmapsto \vec{x}_2$	we add $\mathrm{conv}(\vec{x}_1, \vec{x}_2)$ to the complex,
$\vdots$	K,

Let $S, S' \in K$:

(1) If S = point :

$\vec{x}_j$

$\vec{x}_i$

$S' =$

$S = conv(\vec{x}_i)$ $\vec{x}_k$ $conv(\vec{x}_j, \vec{x}_k)$

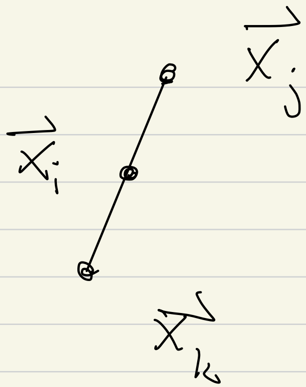
$= \{\vec{x}_i\}$

We want $S \cap S' = \emptyset$!

if not

$$\vec{x}_i = \alpha \vec{x}_j + (1-\alpha)\vec{x}_k$$

$$(0 \leq \alpha \leq 1)$$



so

$$0 = \alpha(\vec{x}_j - \vec{x}_i) + (1-\alpha)(\vec{x}_k - \vec{x}_i)$$

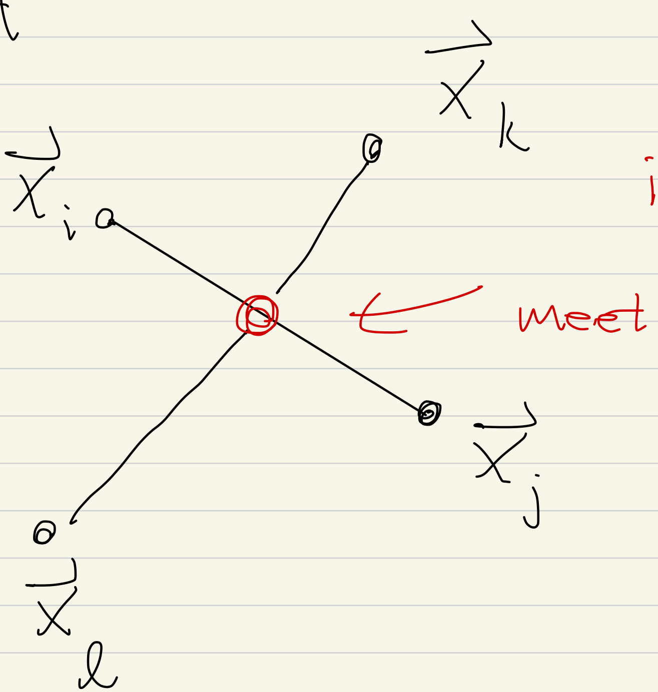
means that

$$\vec{x}_j - \vec{x}_i, \quad \vec{x}_k - \vec{x}_i$$

are linearly dependent, so

$\vec{x}_i, \vec{x}_j, \vec{x}_k$ are not in general position.

(2) If



if

$$\alpha \vec{x}_i + (1-\alpha)\vec{x}_j$$

$$= \beta \vec{x}_k + (1-\beta)\vec{x}_l$$

So $-\vec{x}_i$ both sides!

$$\alpha(\vec{x}_i - \vec{x}_i) + (1-\alpha)(\vec{x}_j - \vec{x}_i) =$$

$$\beta(\vec{x}_k - \vec{x}_i) + (1-\beta)(\vec{x}_\ell - \vec{x}_i)$$

$$(0 \leq \alpha, \beta \leq 1)$$

$\Longrightarrow$

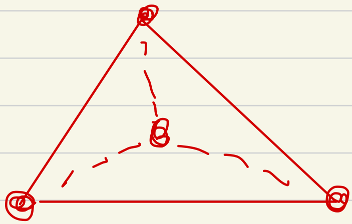
$$(1-\alpha)(\vec{x}_j - \vec{x}_i) + (-\beta)(\vec{x}_k - \vec{x}_i)$$

$$+ (-(1-\beta))(\vec{x}_\ell - \vec{x}_i) = 0$$

$\Longrightarrow$

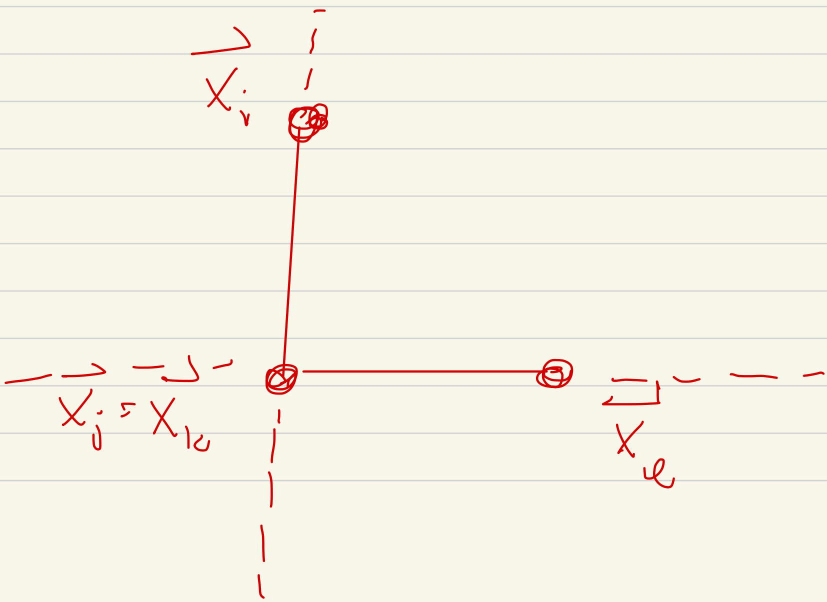
$\vec{x}_i, \vec{x}_j, \vec{x}_k, \vec{x}_\ell$ are not in general position ...

General position $\vec{x}_i, \vec{x}_j, \vec{x}_k, \vec{x}_\ell$



tetrahedron

Say S, S' are lines, but share a vertex



then

$$S \cap S' = \{ \vec{x}_j = \vec{x}_k \}$$

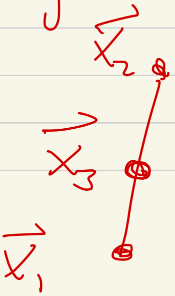
as long as

$\vec{x}_i$, $\vec{x}_j = \vec{x}_k$, $\vec{x}_\ell$

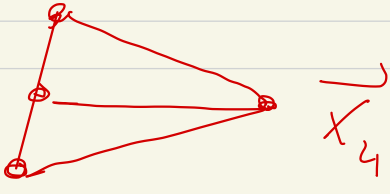
are in general position

Rem: if $\vec{x}_1, \vec{x}_2, \vec{x}_3$ not in

general position:



then $\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4$



Also

$$\vec{x}_i \neq \vec{x}_j$$

unless $\vec{x}_i = \vec{x}_j$

$$\Rightarrow \quad \vec{x}_j - \vec{x}_i = 0$$

so $\vec{x}_i, \vec{x}_2$ are not in general position.

Claim: K built from $G = (V, E)$

$V \to \{\vec{x}_1, \ldots, \vec{x}_n\} \in \mathbb{R}^3$ is a

simplicial complex iff

$\vec{x}_i, \vec{x}_j$ general pos (i, j distinct)

$\vec{x}_i, \vec{x}_j, \vec{x}_k$ " " (i, j, k ")

$\vec{x}_i, \vec{x}_j, \vec{x}_k, \vec{x}_l$ " " (i, j, k, l ")

Claim!

$$\vec{x}_1 = (1, 1, 1)$$

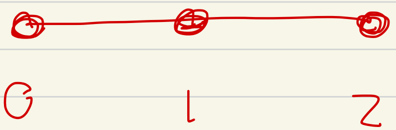
$$\vec{x}_2 = (2, 4, 8)$$

$$\vec{x}_3 = (3, 9, 27) \quad \in \mathbb{R}^3$$

$$\vdots$$

$$\vec{x}_n = (n, n^2, n^3)$$

Claim: With $\vec{x}_i = (i, i^2, i^3)$, any 2, 3, 4 of $\vec{x}_1, \dots, \vec{x}_n$ are in general position.



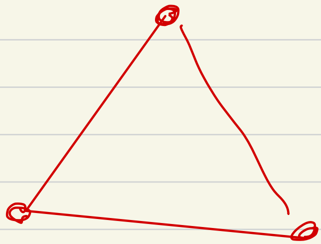
$\subset \mathbb{R}$

$$K = \{ \emptyset, \{0\}, \{1\}, \{2\}, [0,1], [0,2], [1,2] \}$$

$$[0,2] \cap [0,1] = [0,1] \leftarrow \text{not a face of } [0,2]$$

If

$$K = \{ \emptyset, \{\vec{x}_0\}, \{\vec{x}_1\}, \{\vec{x}_2\}, \mathrm{conv}(\vec{x}_0, \vec{x}_1), \mathrm{conv}(\vec{x}_1, \vec{x}_2), \mathrm{conv}(\vec{x}_0, \vec{x}_2) \}$$



triangle

abstract complex :

$$K_{abs} = \{ \emptyset, \{\vec{x}_0\}, \{\vec{x}_1\}, \{\vec{x}_2\}, \{\vec{x}_0, \vec{x}_1\}, \{\vec{x}_1, \vec{x}_2\}, \{\vec{x}_0, \vec{x}_2\} \}$$

$$= \emptyset \cup \vec{V} \cup E$$

$$=$$

Simplicial homology:

- graphs