CPSC 531F Jan 15, 2025

Today, Friday:

- $\beta_0(G) = \dim(H_0(G)) =$ # of connected components of G.

- $\beta_0(G) - \beta_1(G) = \chi(G) = |V| - |E|$

- $\beta_1(\text{Forest}) = 0$ and $\beta_1(G) =$ min # edges needed to remove from G to get a forest

---

- $\partial_1(G) =$ the usual "incidence matrix"

- Simplicial homology for $\dim(K_{abs}) \geq 2$.

Admin:

(1) We now have a piazza page.

(2) I'll hold some office hour(s); if these don't work for you, please email me for an appointment:

jf@cs.ubc.ca

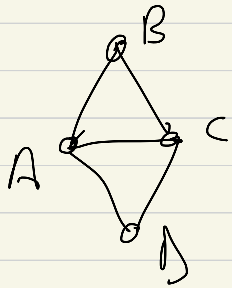
Subject: CPSC 531F etc.

(3) Classes will be recorded on Zoom via Canvas

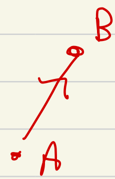
Last time $\partial_1 : \mathcal{C}_1(G) \to \mathcal{C}_0(G)$

"boundary map"

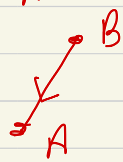
G:



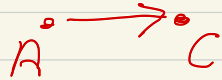
$$\partial_1([A,B]) = [B] - [A]$$



$$\partial_1([B,A]) = [A] - [B]$$



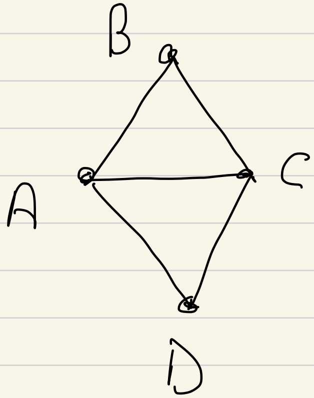
$$\partial_1([A,C]) = [C] - [A]$$



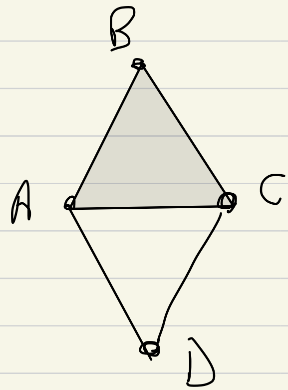
$$\left.\begin{aligned} H_1(G) &= \ker(\partial_1) \\ H_0(G) &= \operatorname{coker}(\partial_1) \end{aligned}\right\} \text{ want to understand}$$

But what if $\dim(K_{abs}) \geq 2$???

Specifically:



vs



Graph

Graph + $\{A, B, C\}$

"2-simplex"

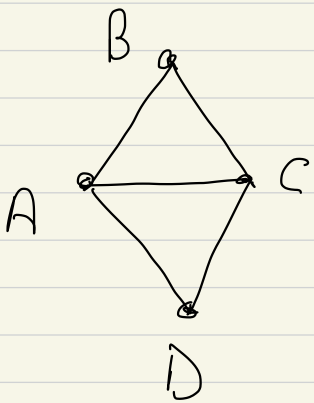
"dim 2 face"

"2-face"

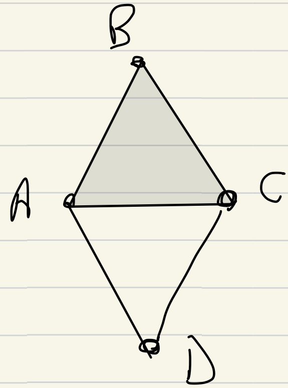
Common in the literature

$\rightsquigarrow$ set of size 3

What is the difference?

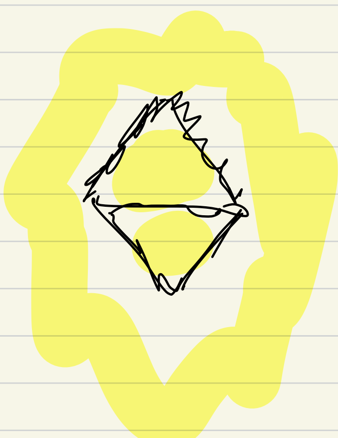


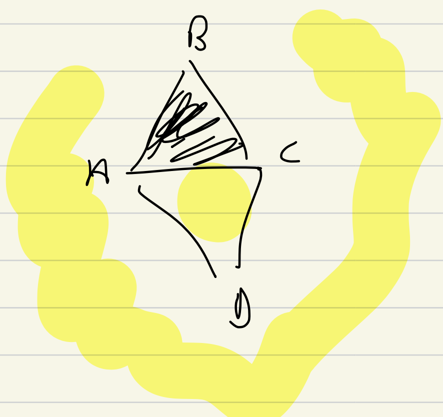
vs



For planar graphs + 2-dim complex

Intuition: $\mathbb{R}^2 \setminus$ simplicial complex



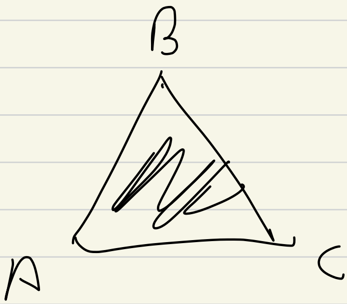


2 "1-holes"

1 "1-hole"

Formally:

$$\{A, B, C\} \in K_{abs}$$

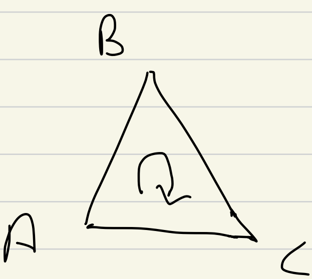


∂_2 !

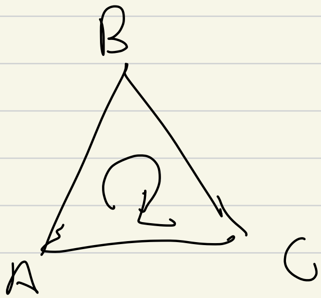
$\uparrow$

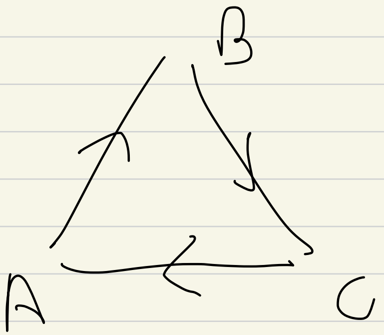
boundary

$$\underbrace{\text{formal } \mathbb{R}\text{-lin combinations of 2-faces}}_{\text{2-forms}} \longrightarrow \text{1-forms}$$



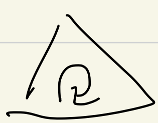
$$[A, B, C] = [B, C, A]$$
$$= [C, A, B]$$

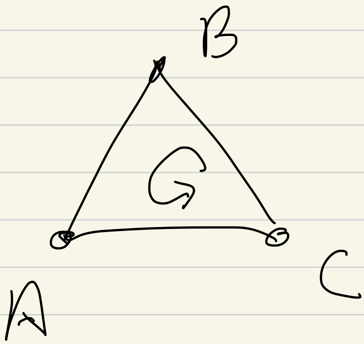
$\partial_2\Big($  $\Big) =$

 $=$

$$[A,B] + [B,C] + [C,A]$$

$$= [A,B] - [A,C] + [B,C]$$

$\partial_1 \partial_2\Big($  $\Big) = 0 \quad (!!)$



$$= -[A,B,C]$$

$$[B,A,C]$$

$$= [A,C,B]$$

$$= [C,B,A]$$

Note

$$\partial_2 [B,A,C]$$

$$\text{rule} = \underbrace{[B,A] + [A,C] + [C,B]}_{1\text{-form}}$$

$$= -\partial_2 [A,B,C]$$

$$\partial_2 [B, A, C]$$

$$= [A, C] + [C, B] + [B, A]$$

$$= [\hat{B}, A, C] - [B, \hat{A}, C]$$

omit B

$$+ [B, A, \hat{C}]$$

i - forms

$$\partial_i \; [u_0, u_1, \ldots, u_i]$$ the i^{th} dim boundary map

$$\overset{\text{def}}{=} \sum_{j=0}^{i} [u_0, \ldots, \hat{u}_j, \ldots, u_i](-1)^j$$

Test

$$\partial_1 [A, B] = [B] - [A]$$

$$\downarrow$$

$$[\hat{A}, B](+1) + [A, \hat{B}](-1)$$

Now: If K_{abs} on V,

so K_{abs} is just a collection of subsets of V,

$\dim(K_{abs}) = 2$, then

$$0 \to \mathcal{C}_2 \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \to 0$$

0-form

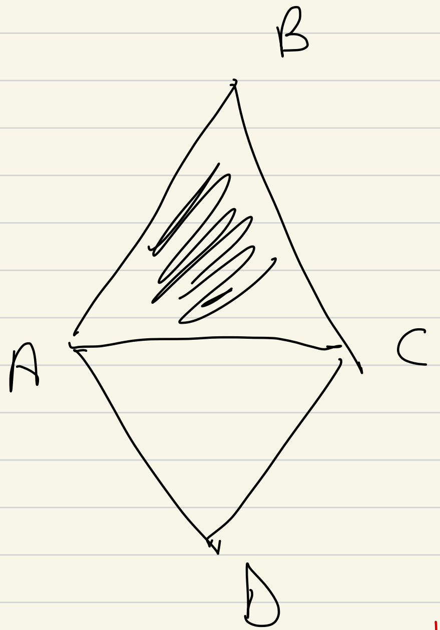
then

$$H_1^{simp}(K_{abs}) = \ker(\partial_1) / \operatorname{im}(\partial_2)$$

$$= Z_1(K_{abs}) \Big/ B_1(K_{abs})$$

$\leftarrow$ "1-cycles"

$B_1(K_{abs}) =$ 1-cycles that are boundaries of 2-dim simp



$$0 \xrightarrow{0} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{0} 0$$

$$H_2(K_{abs}) = \ker\partial_2 \Big/ \mathrm{im}(0)$$

$$= \ker \partial_2$$

$$H_1(K_{abs}) = \ker\partial_1 \Big/ \mathrm{im}(\partial_2)$$

$$H_0(K_{abs}) = \ker 0 \Big/ \mathrm{im}\,\partial_1 = C_0 \Big/ \mathrm{im}\,\partial_1$$

$$= \mathrm{coker}(\partial_1)$$

$$\mathcal{C}_2 \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0$$

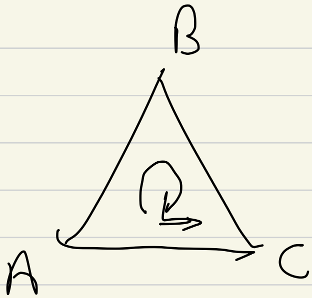
$\mathcal{C}_2$	$\mathcal{C}_1$	$\mathcal{C}_0$
	$[A, B]$	$[A]$
$[A, B, C]$	$[B, C]$	$[B]$
	$[C, D]$	$[C]$
	$[A, D]$	$[D]$
	$[A, C]$	

$$H_i(\mathcal{C}) = \ker \partial_i / \operatorname{im}(\partial_{i+1})$$

This is what we want

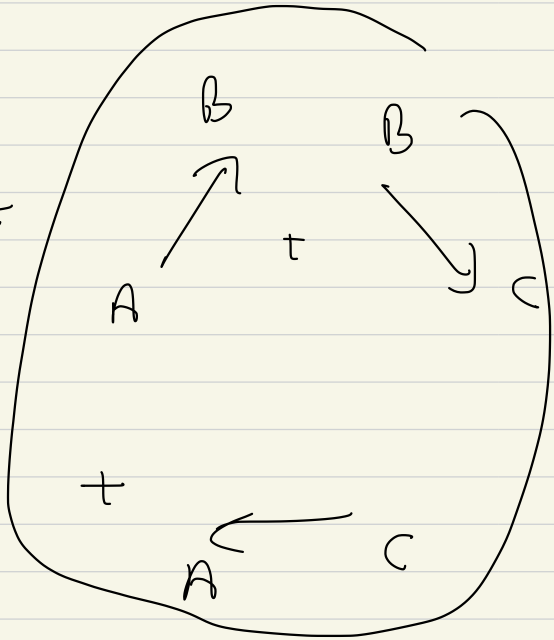
$$\partial_1 \underbrace{\partial_2 \tau}_{} = 0$$

image of ∂_2 $\subset$ kernel of ∂_1



$[A, B, C]$

$\partial_2(\quad) =$



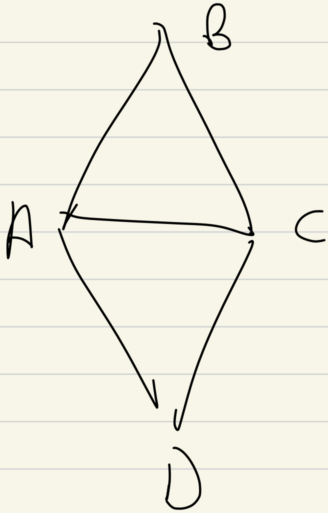
$\partial_1(\quad) = \partial_1(\quad)$

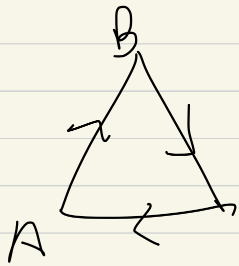
$= (B - A) + (C - B) + (A - C)$

$= B - A + C - B + A - C = 0$

$$\mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0$$

Graph (w/o )

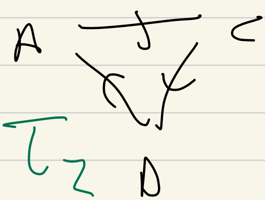




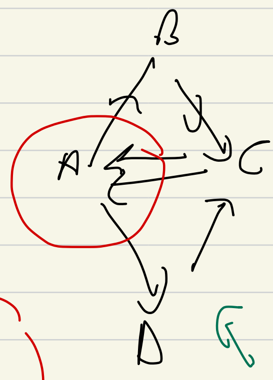
τ_1

cycle

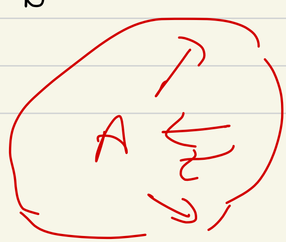
$\in \ker(\partial_1)$



τ_2

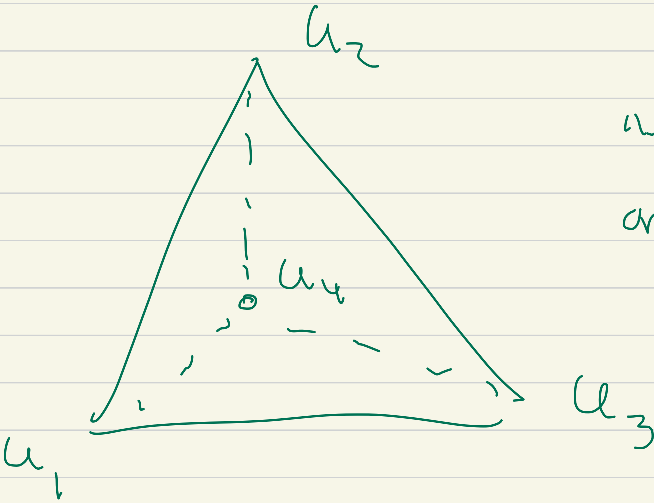


τ_1



$\tau_1 - \tau_2$

$\dim(K_{<4}) = 3$



in $\mathbb{R}^3$

or $\mathbb{R}^N$ $N \geq 3$

$$\partial_2 \left(\partial_3 \, [u_0, u_1, u_2, u_3] \right) = 0$$