CPSC 531F Jan. 20, 2025

- The tedious computation of

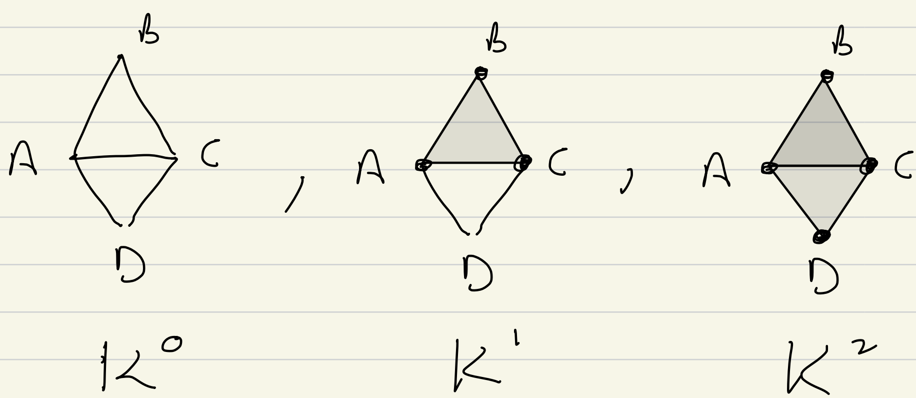
(1) $H_0(G)$, $H_1(G)$ where $G =$ "K_4"

- The tedious computation of

(2) $H_i(K)$ where $K = \text{Power}(\{A,B,C\})$

- The tedious computation of $\beta_i = \dim(H_i(K))$ for (Exercise 6)

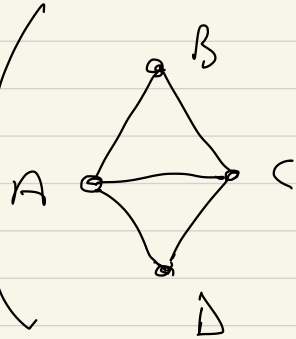
(3)



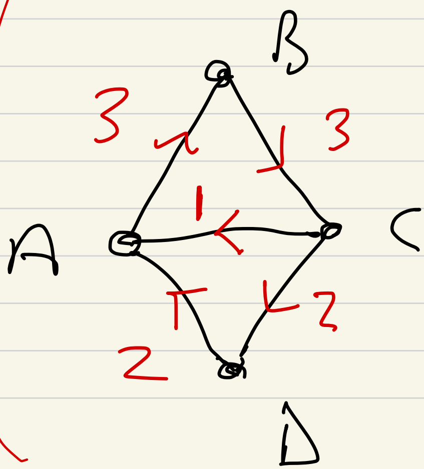
- The interesting Exercise 7 that

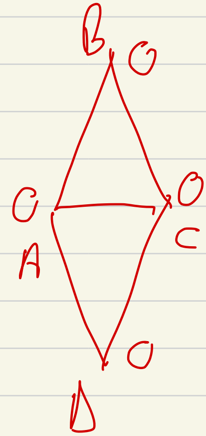
$Z_1(G)$ or $Z_1(K) \overset{\text{def}}{=} \ker(\partial_1)$

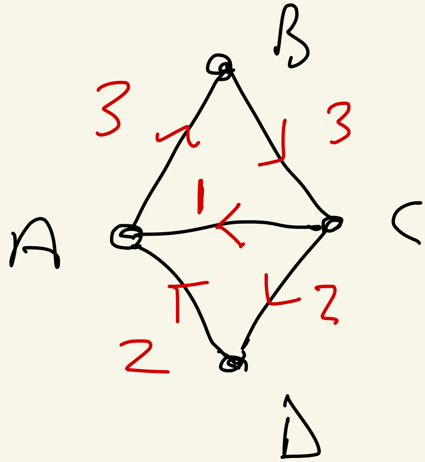
really comes from cycles:

$Z_1\Big($ 

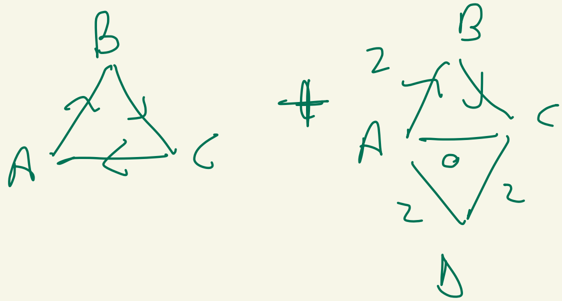
$\Big)$ sample:

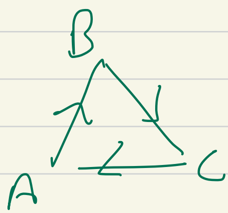
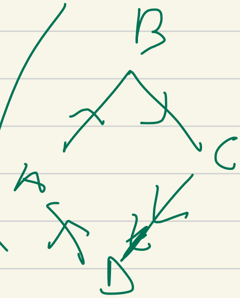
$\partial_1\Big($ 

$\Big) =$ 



$$\partial_1\left(3[A,B] + 3[B,C] + 1[C,A] + 2[C,D] + 2[D,A]\right) = 0\cdot[A] + 0[B] + \dots = 0$$



$=$  $+\ 2\Big($  $\Big)$

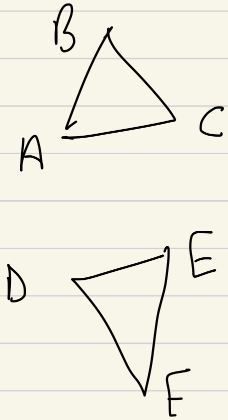
$$= \big([A,B]+[B,C]+[C,A]\big) + 2\big([A,B]+[B,C]+[C,D]+[D,A]\big)$$

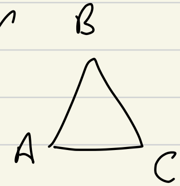
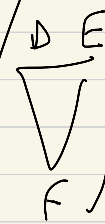
$\in Z_1$, „Zyklus"

cycles

$H_0(G)$, $H_0(K)$ when G or K connected or not; block matrices

G :



$H_i(G) = H_i\Big($  $\Big) \oplus H_i\Big($  $\Big)$

Admin:

Office hours (X561)

M 3:30 - 4:30 pm

Th 2-3 pm

Point of today:

cohomology of differential forms
singular homology

it's clear we need tools
to make computations

Today: Explain why in simplicial
homology, we also want tools

Look at $H_1(G)$, $H_0(G)$ for

G = "K_4" a complete graph

on four vertices!

$G = (V, E)$

$V = \{A, B, C, D\}$

E = all subsets of size 2
of V

$\mathcal{C}_0(G)$ = 0-forms on G

= $\mathbb{R}$-linear combo of

$\underbrace{[A], [B], [C], [D]}_{\text{basis}} \in \mathcal{C}_0(G)$

$\mathcal{C}_1(G)$ = 1-forms on G

= $\mathbb{R}$-linear combo of $[v, v']$

$= -[v', v]$

Basis:

$[A,B], [A,C], [A,D], [B,C], [B,D], [C,D]$

$$\partial_1 : [v, v'] \longmapsto [v'] - [v]$$

$$\partial_1 \Big(\alpha_1 [A,B] + \alpha_2 [A,C] + \_\_ + \alpha_6 [C,D] \Big)$$

$$= \alpha_1 \big([B] - [A] \big) + \alpha_2 \big([C] - [A] \big) + \_\_\_$$

$$\underbrace{\begin{array}{c|cccccc} & [A,B] & [A,C] & [A,D] & [B,C] & [B,D] & [C,D] \\ \hline [A] & -1 & -1 & -1 & 0 & 0 & 0 \\ [B] & 1 & 0 & 0 & -1 & -1 & 0 \\ [C] & 0 & 1 & 0 & 1 & 0 & -1 \\ [D] & 0 & 0 & 1 & 0 & 1 & 1 \end{array}}_{\text{"incidence matrix of } G\text{"}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix}$$

If ∂_G = incidence matrix

then

$$\partial_G \partial_G^T$$

= "Graph Laplacian" = Δ_G

= $D_G - A_G$

diagonal matrix, whose (v,v)-element = "degree of v in G"

adjacency matrix

for

$$\partial_G = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

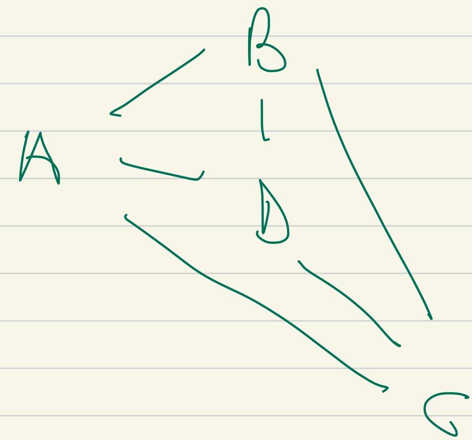
4×6 ;

$$\partial_G \partial_G^T \longrightarrow$$

$4 \times 6 \quad 6 \times 4 \qquad 4 \times 4$

$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} = 3I - A_G$$

G :



$$\partial_1 : C_1 \longrightarrow C_0$$

here $G = K_4$

$$C_1 \cong \mathbb{R}^6, \qquad C_0 \cong \mathbb{R}^4$$

$$H_1(G) \overset{\text{def}}{=} \ker(\partial_1)$$

$$= a_1 [A,B] + a_2 (A,C) + \dots + \alpha_6 [C,D]$$

s.t. $\quad \left(\ker(\partial_1) = \text{nullspace}(\partial_1) \right)$

$$\begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Big\}$ Gaussian elim

row reduced echelon form

or $\quad \Big\}$ ad hoc row ops

move rows 2,3,4 $\to$ 1,2,3

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

add row 1,2,3 to 4

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So

$\ker(\partial_1) = Z_1$ "1-forms that are "cycles""

$$\begin{array}{cl} \left[\begin{array}{c} \alpha_4 + \alpha_5 \\ -\alpha_4 + \alpha_6 \\ -\alpha_5 - \alpha_6 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{array}\right] & \begin{array}{l} \leftarrow [A,B] \\ \leftarrow [A,C] \\ \leftarrow [A,D] \\ \leftarrow [B,C] \\ \\ \\ \end{array} \end{array}$$

$$[B,C]\,\alpha_4 + [A,B]\,\alpha_4 + [A,C](-\alpha_4)$$

$$\xrightarrow{\partial_1} 0$$

$$\downarrow$$

$$\alpha_4\big([A,B] - [A,C] + [B,C]\big)$$

$M =$ represented ∂_1 wrt some choice of $e_0(G)$ (vertices)

$e_1(G)$ (oriented edges)

$$M = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

and

$\ker(M)$ is 3-dim

∂_1 as a matrix!

$$\mathbb{R}^6 \xrightarrow{M} \mathbb{R}^4$$

$\ker(M) = \text{nullspace}(M)$ dim 3

$$\text{rank}(M) = 6 - \dim(\ker)$$
$$= 3$$

$$\text{coker}(M) = \mathbb{R}^4 / \text{Image}(M)$$

$$\dim = 4 - \dim(\text{Image}(M))$$
$$= 4 - 3 = 1$$

So...

$$H_1(G) \stackrel{def}{=} \ker(\partial_1) \quad \text{is } 3\text{-dimensional}$$

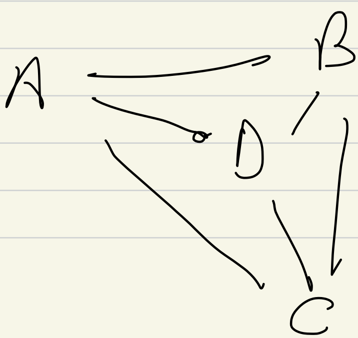
$$H_0(G) \stackrel{def}{=} \operatorname{coker}(\partial_1)$$

$$\stackrel{def}{=} e_0(G) \big/ \mathrm{Image}(\partial_1)$$

is 1-dimensional.

$$b_i(G) = \dim(H_i(G))$$

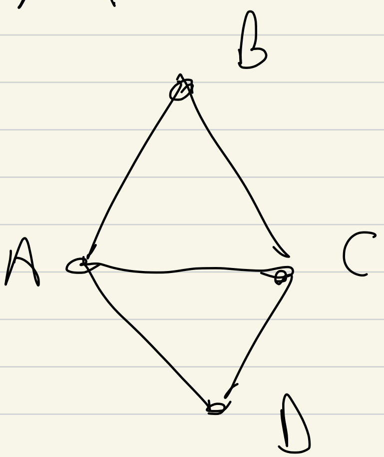
G



Exercise 6!

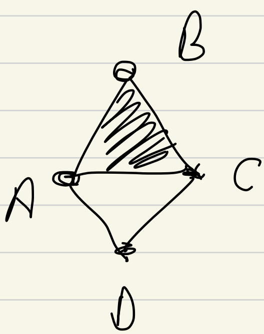
Compute

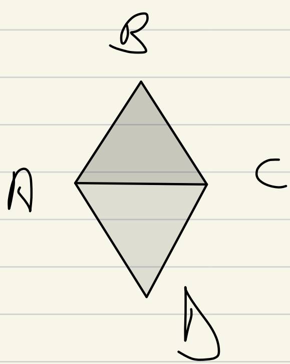
H_0, H_1



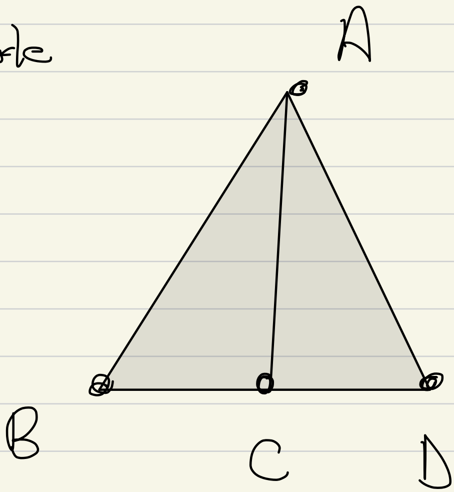
2^{nd}

H_0, H_1, H_2





Note



$$= \mathrm{Cone}_A \left(\underset{B}{\bullet} \text{---} \underset{C}{\bullet} \text{---} \underset{D}{\bullet} \right)$$

Class Ends

$$\partial_1\left([A,B] - [A,C] + [B,C]\right)$$

$$= 0$$

also

$$\partial_2[A,B,C]$$

$$= [B,C] - [A,C] + [A,B]$$

$\partial_1 \partial_2$ anything