CPSC 531F   Jan 22, 2025

- $H_i^{simp}(\underbrace{\langle \{A,B,C\} \rangle}_{\text{2-simplex}})$

- $H_0^{simp}(G)$, G connected

- $b_i^{simp}(G) \overset{def}{=} \dim\left(H_i^{simp}(G)\right)$

- $H_0^{simp}(K)$, $K = K_{abs}$ connected

- Theorem: If $L_{abs} = \mathrm{Cone}_p(K_{abs})$,

then

$$H_i^{simp}(L_{abs}) \simeq \begin{cases} \mathbb{R}, & i = 0 \\ 0, & i \geq 1 \end{cases}$$

$\langle \{A, B, C\} \rangle$ = simplicial complex

generated by $\{A, B, C\}$

$$= \{ \emptyset, \{A\}, \{B\}, \ldots, \{A, B, C\} \}$$

$$= \text{Power}(\{A, B, C\}) = K_{abs}$$

$$0 \to C_2(K_{abs}) \xrightarrow{\partial_2} C_1(K_{abs}) \xrightarrow{\partial_1} C_0(K_{abs}) \to 0$$

bases

$$\overbrace{[A, B, C]}^{B_2} \qquad \overbrace{\boxed{\begin{matrix}[A, B], \\ [A, C] \\ [B, C]\end{matrix}}}^{B_1} \qquad \overbrace{\boxed{[A], [B], [C]}}^{B_0}$$

$$0 \xrightarrow{\partial_3} \mathbb{R} \xrightarrow{\partial_2} \mathbb{R}^3 \xrightarrow{\partial_1} \mathbb{R}^3 \xrightarrow{\partial_0} 0$$

$$H_i(K) = \ker(\partial_i) \big/ \mathrm{Image}(\partial_{i+1})$$

$$\ker(\partial_i) \subset C_i(K)$$

since $\quad \partial_i \partial_{i+1} = 0$

$$\Rightarrow \quad \mathrm{Image}(\partial_{i+1}) \subset \ker(\partial_i)$$

$$H_2(K) = \ker(\partial_2) \big/ \mathrm{Im}(\partial_3)$$

$$= \ker(\partial_2)$$

$$\partial_2 = \begin{array}{cc} & [A,B,C] \\ \begin{matrix} {[A,B]} \\ {[A,C]} \\ {[B,C]} \end{matrix} & \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \end{array}$$

$$\partial_2 \overbrace{[A,B,C]}^{\text{all of } C_2(K) \text{ is gen by one elt}}$$

$$\overset{\text{def}}{=} [\hat{A},B,C] - [A,\hat{B},C] + [A,B,\hat{C}]$$

$$= [B,C] - [A,C] + [A,B]$$

is non-zero.

So $H_2(K) = \ker(\partial_2) = 0$

$$\partial_1 : \begin{array}{c c} & \begin{array}{ccc} [A,B] & [A,C] & [B,C] \end{array} \\ \begin{array}{c} [A] \\ [B] \\ [C] \end{array} & \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$\partial_1 [A, B] = [B] - [A]$$

$\vdots$

$\ker(\partial_1)$:

$$\ker \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}_{B_0, B_1}$$

"cheat"

$$= \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

free

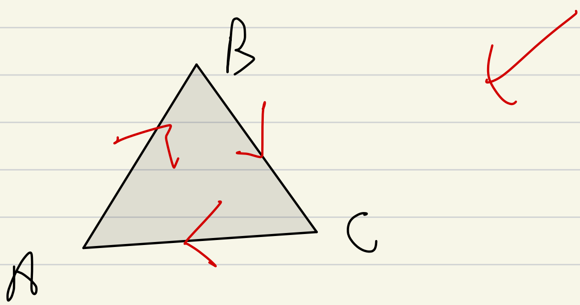
$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \;\middle|\; \begin{matrix} \alpha_1 = \alpha_3 \\ \alpha_2 = -\alpha_3 \end{matrix} \right\}$$

$$= \left\{ \alpha_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

B_1

$$= \left\{ \alpha_3 \left([A,B] - [A,C] + [B,C] \right) \right\}$$



So: $C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1}$

$$\ker(\partial_1) = \operatorname{Span}\big([A,B] - [A,C] + [B,C]\big)$$

$$= \operatorname{Image}(\partial_2)$$

$$H_1^{simp}(K_{abs}) = \ker(\partial_1) / \operatorname{Im}(\partial_2) = 0$$

Also

$$H_0^{simp}(K_{abs}) = \ker(\partial_0) / \operatorname{Im}(\partial_1)$$

$$C_0(G) \xrightarrow{\partial_0} 0 \qquad = C_0(G) / \operatorname{Im}(\partial_1)$$

$$\partial_1 [A,B] = [B] - [A]$$

$$\partial_1 [A,C] = [C] - [A]$$

$$\partial_1 [B,C] = [C] - [B]$$

$$H_0^{simp}(K) = H_0^{simp}(G)$$

where $G = (V, E)$

$$\left(\begin{array}{l} V = \{ v \mid \{v\} \in K \} \\ E = \{ \{v, v'\} \mid \{v, v'\} \in K \} \end{array} \right)$$

We know

$$\beta_1 [A] + \beta_2 [B] + \beta_3 [C]$$

modulo $\text{Image}(\partial_1)$

$\mathcal{B}_0$

vll

$\text{Image}(\partial_2) = \text{Span}\left\{ [B]-[A], [C]-[A] \right\}$

$$[B]-[C] \equiv 0 \text{ modulo } \mathcal{B}_0$$

$$[B] \equiv [C] \text{ modulo } \mathcal{B}_0$$

$$[A] \equiv [C] \text{ modulo } \mathcal{B}_0$$

$$[A] \equiv [B] \text{ modulo } \mathcal{B}_0$$

$$\left.\begin{aligned} &\beta_1[A] + \beta_2[B] + \beta_3[C] \\ &\equiv (\beta_1+\beta_2+\beta_3)[A] \\ &\equiv (\beta_1+\beta_2+\beta_3)[B] \end{aligned}\right\} \text{ modulo } \mathcal{B}_0$$

$$[A] \equiv [B] \mod B_0$$

$$[A] + B_0 = [B] + B_0$$

$$\shortparallel$$

$$\{[A] + \tau \mid \tau \in B_0\}$$

Rem: If $U \subset W$ vector spaces over $\mathbb{R}$,

$$\dim(W/U) = \dim(W) - \dim(U)$$

e.g.,

$$\mathbb{R}^3 \xrightarrow{\partial_1} \mathbb{R}^3$$

$$e_1 \xrightarrow{\partial_1} e_0$$

$$\text{Image}(\partial_1) = \text{Span}\left\{ \begin{array}{l} [B]-[A], \\ [C]-[A] \end{array} \right\}$$

$$= \left\{ \beta_1 [A] + \beta_2 [B] + \beta_3 [C] \;\middle|\; \beta_1 + \beta_2 + \beta_3 = 0 \right\}$$

$$\partial_1 \quad \begin{array}{l} [A,B] = [B]-[A] \\ [A,C] = [C]-[A] \\ \quad \vdots \end{array}$$

$$\mathbb{R}^3 \Big/ \operatorname{im}(\partial_1) \longleftrightarrow B_0$$

$$\mathbb{R}^3 \Big/ \left\{ \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \,\middle|\, \beta_1 + \beta_2 + \beta_3 = 0 \right\}$$

As a vector space, this is 1-dimensional, generated by $[A]$'s image in $\mathbb{R}^3/B_0$

$[A] + B_0$,

OR

$(2A) + B_0$, OR $2[A] - 3[C] + B_0$

$\mathbb{Z}/3\mathbb{Z}$ math "numbers modulo 3"

$3\mathbb{Z} < \mathbb{Z}$ under $+$

$\parallel$

$\{\ldots, -3, 0, 3, 6, \ldots\}$

Def: $\mathbb{Z}/3\mathbb{Z}$ = the set of "$3\mathbb{Z}$-cosets":

$0 + 3\mathbb{Z} = \{\ldots, -3, 0, 3, 6, \ldots\}$

$4+3\mathbb{Z} = 1 + 3\mathbb{Z} = \{\ldots, -2, 1, 4, 7, \ldots\}$

$2 + 3\mathbb{Z} = \{\ldots, -1, 2, 5, 8, \ldots\}$

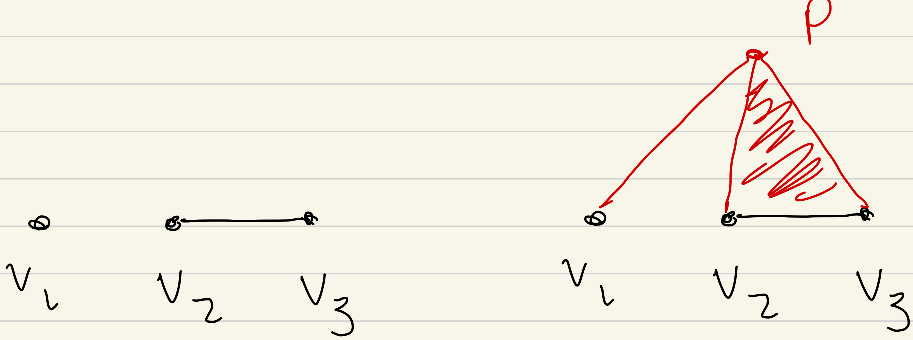
$= 5+3\mathbb{Z} = -4+3\mathbb{Z}$

$2 \equiv 5 \pmod 3$

means

$2 + 3\mathbb{Z} = 5 + 3\mathbb{Z}$

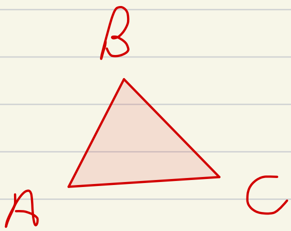
Let K_{abs} be an abs. simp. compl.
on vertex set $V = \{ v \mid \{v\} \in K_{abs} \}$
let $p \notin V$.

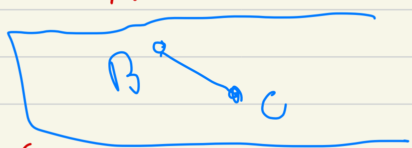


$$\mathrm{Cone}_p(K_{abs}) = \bigcup_{A \subset K_{abs}} \{ A, A \cup \{p\} \}$$

Thm: $H_i^{simp}(\mathrm{Cone}_p(K_{abs})) \cong$

$$\begin{cases} \mathbb{R} & \text{if } i = 0 \\ 0 & \text{if } i \geq 1 \end{cases}$$

Note  $= \mathrm{Cone}_A \langle \{B, C\} \rangle$



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$\mathrm{Cone}_B(\{\emptyset, \{C\}\})$

$\mathrm{Cone}_A\left(\mathrm{Cone}_B(\{\emptyset, \{C\}\})\right)$

<u>Pf:</u> $i \geq 1$ and

$$\mathcal{C}_{i+1} \xrightarrow{\partial_{i+1}} \mathcal{C}_i \xrightarrow{\partial_i} \mathcal{C}_{i-1}$$

(1) Let $\tau \in C_i(K)$:

claim: there is a τ'

s.t. $\tau' \equiv \tau \mod (\text{Image}(\partial_{i+1}))$

s.t.

τ' ~~always~~ only has elements

of the form constant times

$[P, u_{\cancel{0}}, \ldots, u_i]$

e_i: $[u_0, u_1, \ldots, u_i]$

(2) If such a τ', $\partial_i \tau' = 0 \Rightarrow \tau' = 0$