CPSC 531F Jan 27, 2024

- Last time:

$$H_i(\text{Any Cone}) \simeq \begin{cases} 0 & \text{if } i \geq 1 \\ \mathbb{R} & \text{if } i = 0 \end{cases}$$

- Euler characteristic

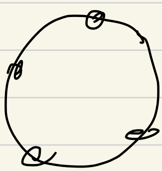
- Mayer-Vietoris:

$$\left.\begin{array}{l} G = G_1 \cup G_2 \\ \text{OR} \\ K = K_1 \cup K_2 \end{array}\right\} \text{same argument}$$

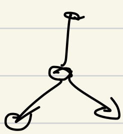
$G = (V, E)$

$\chi(G)$ = Euler characteristic of $G$

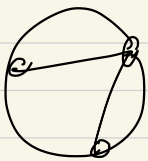
$\stackrel{\text{def}}{=} |V| - |E|$

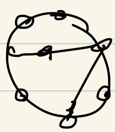
$\chi\Big($  $\Big) = 0$

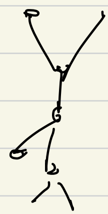
cycle

$\chi\Big($  $\Big) = 1$

$$\chi(G) = |V| - |E|$$

$\chi\Big($  $\Big) = -2$

$\chi\Big($  $\Big) = -2$

$\chi\Big($    $\Big)$

$$= \chi(\ ) + \chi(\ ) + \chi(\ )$$

$$-2 + 0 + 1$$

$$= -1$$

$H_0, H_1$ graph : defined

$$C_1(G) \to C_0(G)$$

$$\partial_1 : \mathbb{R}^{|E|} \to \mathbb{R}^{|V|}$$

$$H_1(G) = \ker(\partial_1)$$

$$H_0(G) = \operatorname{coker}(\partial_1)$$

$$= C_0(G) / \operatorname{image}(\partial_1)$$

Claim:

$$\underbrace{\dim(H_0(G))}_{\beta_0(G)} - \underbrace{\dim(H_1(G))}_{\beta_1(G)}$$

$$\underbrace{\beta_0(G)} - \beta_1(G) =$$

$$\Big(\dim C_0(G) - \dim(\operatorname{Im}(\partial_1))\Big)$$

$$\shortparallel$$

$$- \Big(\dim(C_1(G)) - \operatorname{Rank}(\partial_1)\Big)$$

$$= \dim C_0 - \dim(C_1)$$

$$= |V| - |E| = \chi(G)$$

More generally (EXERCISE): If

$$\mathcal{C}_d(K) \xrightarrow{\partial_d} \mathcal{C}_{d-1}(K) \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_2} \mathcal{C}_1(K) \xrightarrow{\partial_1} \mathcal{C}_0(K)$$

$$\chi(\mathcal{C}_d \to \cdots \to \mathcal{C}_0)$$

$$= \dim(\mathcal{C}_0) - \dim(\mathcal{C}_1) + \dim(\mathcal{C}_2) - \dim(\mathcal{C}_3) + \cdots$$

EXERCISE

$$= \beta_0 - \beta_1 + \beta_2 - \beta_3 + \cdots$$

$$\beta_i(K) = \dim H_i(K) = \dim \frac{Z_i(K)}{B_i(K)}$$

$$Z_i(K) = \ker(\partial_i)$$

$$B_i(K) = \text{Image}(\partial_{i+1})$$

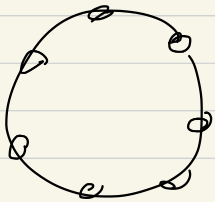
(All we really use is that

$$\partial_i \partial_{i+1} = 0$$
)

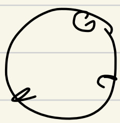
In particular

$$\beta_0(G) - \beta_1(G) = |V| - |E|$$

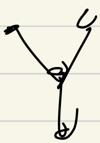
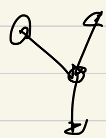
$$= \chi(G)$$

$\chi\Big($  $\Big) = 0$

$= |V| - |E|$

$\beta_0\Big($  $\Big) = 1$

$\beta_1\Big($  $\Big) = 1$

$\beta_0\Big($  $\Big) = 1, \quad \beta_1\Big($  $\Big) = 0$

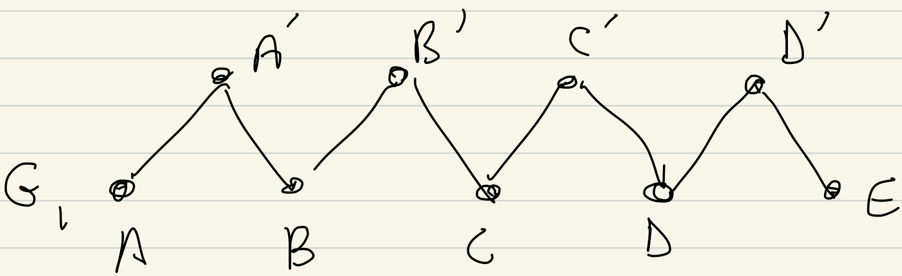
Mayer-Vietoris sequence:

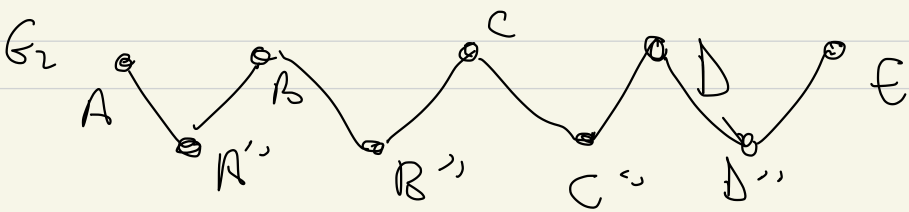
Say that $G$ is a graph

($K$ is an abs simp complex)

and $G_1, G_2 \subset G$ s.t. $G_1 \cup G_2 = G$.

($K_1, K_2 \subset K$ s.t. $K_1 \cup K_2 = K$).

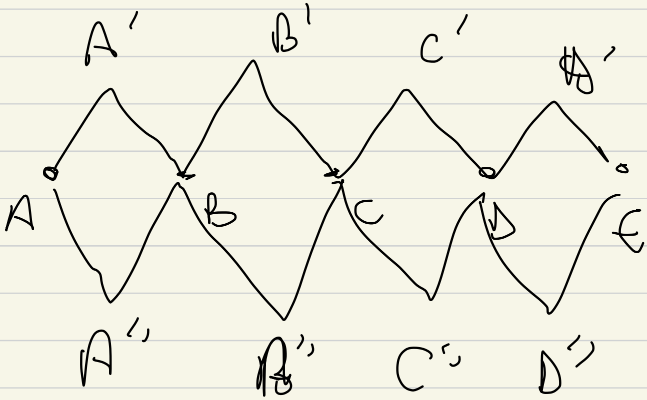


In this example:

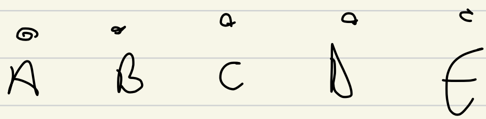
$G_1, G_2$ are trees, each $\beta_0 = 1$, $\beta_1 = 0$

$G_1 \cup G_2 = G$




$\beta_0 = 1$, $\beta_1 = 4$

$G_1 \cap G_2$




$b_0 = 5$, $b_1 = 0$

Theorem: There is an <u>exact</u> <u>sequence</u>

$$0 \to H_1(G_1 \cap G_2) \to H_1(G_1) \oplus H_1(G_2) \to H_1(G) \to H_0(G_1 \cap G_2) \to H_0(G_1) \oplus H_0(G_2) \to H_0(G) \to 0$$

$$\begin{array}{c} H_1(G) \\ \wr\!\wr \end{array}$$

$$0 \to 0 \to 0 \oplus 0 \to \mathbb{R}^4 \to \mathbb{R}^5 \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \to 0$$

$H_0(G_1 \cap G_2)$ $\quad \uparrow \nearrow \nearrow$ $\quad H_0$

Exact sequence

$$V_0 \leftarrow V_1 \leftarrow V_2 \leftarrow V_3 \cdots$$

where $\xleftarrow{\partial_i} V_i \xleftarrow{\partial_{i+1}} V_{i+1}$

$$\ker(\partial_i) = \operatorname{Im}(\partial_{i+1})$$