

CPSC 531F

Jan 27, 2024

- Last time:

$$H_i(\text{Any Cone}) \cong \begin{cases} 0 & \text{if } i \geq 1 \\ \mathbb{R} & \text{if } i = 0 \end{cases}$$

- Euler characteristic

- Mayer-Vietoris:

$$\begin{array}{l} G = G_1 \cup G_2 \\ \text{OR} \\ K = K_1 \cup K_2 \end{array} \left. \vphantom{\begin{array}{l} G = G_1 \cup G_2 \\ \text{OR} \\ K = K_1 \cup K_2 \end{array}} \right\} \begin{array}{l} \text{same} \\ \text{argument} \end{array}$$

$$G = (V, E)$$

$\chi(G)$ = Euler characteristic
of G

$$\stackrel{\text{def}}{=} |V| - |E|$$

$$\chi \left(\begin{array}{c} \text{cycle} \\ \text{cycle} \end{array} \right) = 0$$

$$\chi \left(\begin{array}{c} \text{star} \\ \text{star} \end{array} \right) = 1$$

$$\chi(G) = |V| - |E|$$

$$\chi(\text{triangle}) = -2$$

$$\chi(\text{square}) = -2$$

$$\chi(\text{triangle} \cup \text{circle} \cup \text{Y-shape}) = \chi(\downarrow) + \chi(\downarrow) + \chi(\downarrow)$$

$$= -2 + 0 + 1$$

$$= -1$$

H_0, H_1 graph : defined

$$e_1(G) \rightarrow e_0(G)$$

$$\partial_1 : \mathbb{R}^{|E|} \rightarrow \mathbb{R}^{|V|}$$

$$H_1(G) = \ker(\partial_1)$$

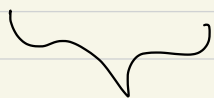
$$H_0(G) = \operatorname{coker}(\partial_1)$$

$$= e_0(G) / \operatorname{image}(\partial_1)$$

Claim: $\beta_0(G) \quad \beta_1(G)$

$$\dim(H_0(G)) - \dim(H_1(G))$$

$$\beta_0(G) - \beta_1(G) =$$



$$(\dim E_0(G) - \dim(\text{Im}(\partial_1)))$$

$$- (\dim(E_1(G)) - \text{Rank}(\partial_1))$$

$$= \dim(E_0) - \dim(E_1)$$

$$= |V| - |E| = \chi(G)$$

More generally (EXERCISE) ! If

$$E_d(K) \xrightarrow{\partial_d} E_{d-1}(K) \xrightarrow{\partial_{d-1}} \dots \xrightarrow{\partial_2} E_1(K) \xrightarrow{\partial_1} E_0(K)$$

$$\chi(E_d \rightarrow \dots \rightarrow E_0)$$

$$= \dim(E_d) - \dim(E_{d-1}) + \dim(E_{d-2}) - \dim(E_{d-3}) + \dots$$

EXERCISE

$$= \beta_0 - \beta_1 + \beta_2 - \beta_3 + \dots$$

$$\beta_i(K) = \dim H_i(K) = \dim \frac{Z_i(K)}{B_i(K)}$$

$$Z_i(K) = \ker(\partial_i)$$

$$B_i(K) = \text{Image}(\partial_{i+1})$$

$$\left(\begin{array}{l} \text{All we really use is that} \\ \partial_i \partial_{i+1} = 0 \end{array} \right)$$

In particular

$$\begin{aligned} \beta_0(G) - \beta_1(G) &= |V| - |E| \\ &= \chi(G) \end{aligned}$$

$$\chi(\text{circle with 6 vertices}) = 0$$

$$= |V| - |E|$$

$$\beta_0(\text{circle}) = 1$$

$$\beta_1(\text{circle}) = 1$$

$$\beta_0(\text{Y-shape}) = 1, \quad \beta_1(\text{Y-shape}) = 0$$

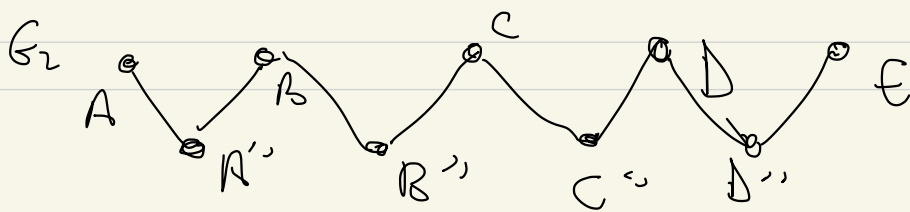
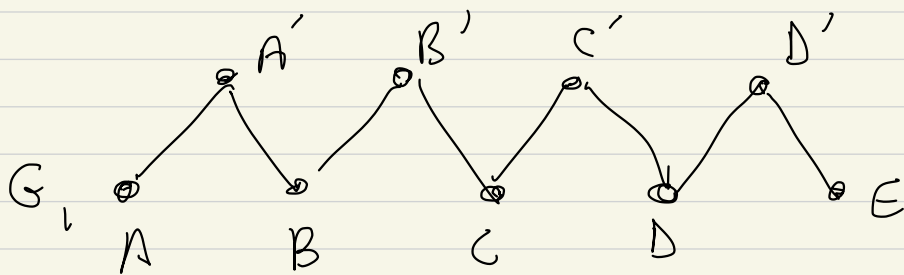
Mayer-Vietoris sequence:

Say that G is a graph

(K is an abs simp complex)

and $G_1, G_2 \subset G$ s.t. $G_1 \cup G_2 = G$.

($K_1, K_2 \subset K$ s.t. $K_1 \cup K_2 = K$).

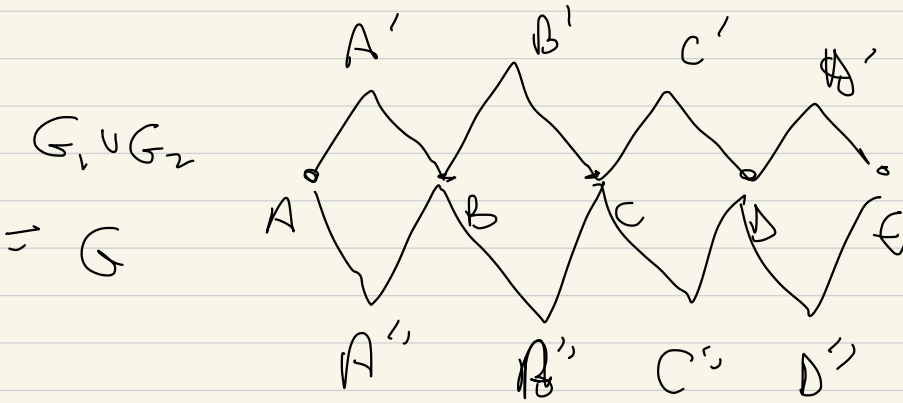


In this example:

G_1, G_2 are trees, each

$$b_0 = 1$$

$$b_1 = 0$$



$$b_0 = 1, \quad b_1 = 4$$

$$G_1 \cap G_2 \quad \overset{\circ}{A} \quad \overset{\circ}{B} \quad \overset{\circ}{C} \quad \overset{\circ}{D} \quad \overset{\circ}{E}$$

$$b_0 = 5, \quad b_1 = 0$$

Theorem: There is an exact sequence

$$0 \rightarrow H_1(G_1 \wedge G_2) \rightarrow H_1(G_1) \oplus H_1(G_2) \rightarrow H_1(G)$$

$$\rightarrow H_0(G_1 \wedge G_2) \rightarrow H_0(G_1) \oplus H_0(G_2) \rightarrow H_0(G)$$

$$\rightarrow 0$$

$$H_1(G)$$

$$0 \rightarrow 0 \rightarrow \mathbb{C} \oplus \mathbb{C} \rightarrow \mathbb{R}^4$$

$$\mathbb{R}^5 \rightarrow \mathbb{R} \oplus \mathbb{R} \rightarrow \mathbb{R} \rightarrow 0$$

$$H_0(G_1 \wedge G_2) \quad \downarrow \quad \uparrow \quad \nearrow$$

$$H_0$$

Exact sequence

$$V_0 \xrightarrow{\partial_0} V_1 \xrightarrow{\partial_1} V_2 \xrightarrow{\partial_2} V_3 \dots$$

where

$$\begin{array}{ccc} & \partial_i & \\ & \xrightarrow{\quad} & \\ & \partial_{i+1} & \\ & \xrightarrow{\quad} & \\ & \partial_{i+2} & \\ & \xrightarrow{\quad} & \end{array} V_i \xrightarrow{\quad} V_{i+1}$$

$$\ker(\partial_i) = \text{Im}(\partial_{i+1})$$