CPSC 531F Jan 29, 2025

Today: $G_1, G_2 \subset G = (V, E)$

s.t. $G_1 \cup G_2 = G$

then there is an exact sequence

$$0 \to H_1(G_1 \cap G_2) \to H_1(G_1) \oplus H_1(G_2) \to H_1(G)$$

$$\xrightarrow{\delta} H_0(G_1 \cap G_2) \to H_0(G_1) \oplus H_0(G_2) \to H_0(G)$$

$$\to 0$$

Exact sequence: a chain is a sequence

$$0 \to U_k \xrightarrow{d_{k+1}} \cdots \xrightarrow{d_3} U_2 \xrightarrow{d_2} U_1 \xrightarrow{d_1} U_0 \to 0$$

We have (1) sequence of vector spaces

$U_0, \ldots, U_k$ of $\mathbb{R}$-vector spaces

"abstract vector spaces", and

(2) maps $d_i : U_i \to U_{i-1}$

(3) $d_i d_{i+1} : U_{i+1} \to U_{i-1}$,

then $d_i d_{i+1} = 0$

E.g. If G is a graph:

then $$0 \xrightarrow{\partial_2} \mathcal{C}_1(G) \xrightarrow{\partial_1} \mathcal{C}_0(G) \xrightarrow{\partial_0} 0$$

is a chain. Also $K = K_{abs}$ is a abstract simplicial complex, of dimension j

$$0 \to \mathcal{C}_j(K) \xrightarrow{\partial_j} \mathcal{C}_{j-1}(K) \to \cdots \xrightarrow{\partial_1} \mathcal{C}_0(K) \to 0$$

and $$\partial_i \partial_{i+1} = 0$$

We use $(\bar{U}_\bullet, d_\bullet)$ to denote

$$0 \to \bar{U}_k \xrightarrow{d_k} \bar{U}_{k-1} \to \cdots \xrightarrow{d_0} \bar{U}_0 \to 0$$

Hence $\forall \tau \in \bar{U}_{i+1}$,

$d_i(d_{i+1}\,T) = 0$ and so

$$d_i(\text{ Image of } d_{i+1}) = 0$$

so

$$\text{Image of } d_{i+1} \subset \ker(d_i).$$

We define

$$H_i((\bar{V}_\bullet, d_\bullet)) \stackrel{\text{define}}{=}$$

$$\ker(d_i) \big/ \text{Image}(d_{i+1}).$$

We say $(\bar{V}_\bullet, d_\bullet)$ is <u>exact</u> in position i if

$$H_i((U_\cdot, d_\cdot)) = 0, \text{ i.e.}$$

$$\mathrm{Image}(d_{i+1}) = \ker(d_i).$$

And $(U_\cdot, d_\cdot)$ is <u>exact</u> if it is exact in every position.

Example: Say that $S \subset T$ are finite sets. Then

$$\mathbb{R}[S] = \text{formal } \mathbb{R}\text{-linear combination of elements of } S$$

$$\mathbb{R}[S] \subset \mathbb{R}[T].$$

$$S = \{A, B\}, \quad T = \{A, B, C\}$$

$$\mathbb{R}[S] = \{\, \alpha_1 A + \alpha_2 B \mid \alpha_1, \alpha_2 \in \mathbb{R} \}$$

so this contains

$$0, \quad 3 \cdot A, \quad 5 \cdot A + (-7) B,$$

$$12 \cdot B, \quad \left(\frac{\pi^2+1}{3}\right) A - (\sqrt{2}) B, \ldots$$

$$\mathbb{R}[T] = \{\, \alpha_1 A + \alpha_2 B + \alpha_3 C \mid \alpha_i \in \mathbb{R} \}$$

and so

$$\mathbb{R}[S] \subset \mathbb{R}[T]$$

We also think of

$$\left(\mathbb{R}[S]\right) \longleftrightarrow \left(\text{map } S \to \mathbb{R}\right)$$

$$5A + (-7)B \longleftrightarrow \begin{array}{l} A \mapsto 5 \\ B \mapsto -7 \end{array}$$

$$\longleftrightarrow \quad \mathbb{R}^S \quad (\text{functions } S \to \mathbb{R})$$

Let S_1, S_2 be finite sets.

We claim there is an exact sequence:

$$0 \to \mathbb{R}[S_1 \cap S_2] \to \mathbb{R}[S_1] \oplus \mathbb{R}[S_2] \to \mathbb{R}[S_1 \cup S_2] \to 0$$

Example:

$$S_1 = \{A, B, C, D\}$$

$$S_2 = \{D, E\}$$

$$S_1 \cap S_2 = \{D\}$$

$$S_1 \cup S_2 = \{A, B, C, D, E\}$$

$$|S_1| + |S_2| = |S_1 \cap S_2| + |S_1 \cup S_2|$$

here

$$\underbrace{|S_1|}_{4} + \underbrace{|S_2|}_{2} = \underbrace{|S_1 \cap S_2|}_{1} + \underbrace{|S_1 \cup S_2|}_{5}$$

So let

$$\mathbb{R}[S_1 \cap S_2] \xrightarrow{\mu} \mathbb{R}[S_1] \oplus \mathbb{R}[S_2]$$

recall:

U, W are $\mathbb{R}$-vector spaces

$$U \oplus W \overset{\text{def}}{=} \{ (u, w) \in u \in U, w \in W \}$$

$$\mathbb{R}^2 \oplus \mathbb{R}^3 \leftarrow \{ (x_1, x_2), (y_1, y_2, y_3) \}$$

$$(x_1, x_2) \quad (y_1, y_2, y_3)$$

$$\{ (x_1, x_2, y_1, y_2, y_3) \mid x_i, y_j \in \mathbb{R} \}$$

$$\mathbb{R}^5 \leftarrow$$

Here

$$\mathbb{R}^2 \oplus \mathbb{R}^2 = \{ (\vec{x}, \vec{y}) \mid \vec{x} \in \mathbb{R}^2, \vec{y} \in \mathbb{R}^2 \}$$

$$= \{ ((x_1, x_2), (y_1, y_2)) \} \simeq \mathbb{R}^4$$

$=$

There is a Kronecker product

$\vec{U} \otimes \vec{W}$ (we don't need for now),

but

$$\dim(\vec{U} \oplus \vec{W}) = \dim(\vec{U}) + \dim(\vec{W})$$

$$\dim(\vec{U} \otimes \vec{W}) = \dim(U) \dim(\vec{W})$$

$$S_1 \cap S_2 \subset S_1$$

$$S_1 \cap S_2 \subset S_2$$

So let

$$\mu : \mathbb{R}[S_1 \cap S_2] \to \mathbb{R}[S_1] \oplus \mathbb{R}[S_2]$$

be

$$\mu(\tau) \mapsto (\tau_1, \tau_2)$$

where τ_1 is τ viewed as in $\mathbb{R}[S_1]$

τ_2 " " " " " $\mathbb{R}[S_2]$

Example $S_1 = \{A, B, C, D\}$, $S_2 = \{D, E\}$, say

$$\tau \in \mathbb{R}[S_1 \cap S_2]$$

for example $\tau = \sqrt{2}\, D$

So $\sqrt{2}\, D$ is also in $\mathbb{R}[S_1]$ and $\mathbb{R}[S_2]$

So

$$\mu(\sqrt{2}D) = (\sqrt{2}D, \sqrt{2}D)$$

Now define

$$\nu : \mathbb{R}[S_1] \oplus \mathbb{R}[S_2] \to \mathbb{R}[S_1 \cup S_2]$$

given by

$$\sigma_1 \in \mathbb{R}[S_1], \quad \sigma_2 \in \mathbb{R}[S_2]$$

then

$$\nu(\sigma_1, \sigma_2) = \tilde{\sigma}_1 \cdot \tilde{\sigma}_2$$

where $\hat{\sigma}_i$ is just σ_i viewed as lying in $\mathbb{R}[S_1 \cup S_2]$.

$$\mathbb{R}[S_1 \cap S_2] \xrightarrow{\mu} \mathbb{R}[S_1] \oplus \mathbb{R}[S_2]$$

e.g.

$$\sqrt{2}\,D \overset{\mu}{\longmapsto} (\sqrt{2}\,D, \sqrt{2}\,D)$$

and

$$\mathbb{R}[S_1] \oplus \mathbb{R}[S_2] \xrightarrow{\nu} \mathbb{R}[S_1 \cup S_2]$$

e.g.

$$(\sqrt{5}\,A + 2B + 4D,\ 7D + \sqrt{6}\,E)$$

$$\overset{\nu}{\longmapsto} (\sqrt{5}\,A + 2B + 4D) - (7D + \sqrt{6}\,E)$$

$$\in \mathbb{R}[S_1 \cup S_2]$$

$$\mathbb{R}[\{A, B, C, D, E\}]$$

also

$$(\sqrt{2}\,D, \sqrt{2}\,D) \overset{\nu}{\longmapsto} (\sqrt{2}\,D) - (\sqrt{2}\,D) = 0$$

So

$$0 \xrightarrow{d_3 = 0} \mathbb{R}[S_1 \cap S_2] \xrightarrow{d_2 = \mu} \mathbb{R}[S_1] \oplus \mathbb{R}[S_2] \xrightarrow{d_1 = \nu} \mathbb{R}[S_1 \cup S_2] \xrightarrow{d_0 = 0} 0$$

claim

$$d_1 \, d_2 \,(\text{anything}) = 0$$

$$\tau \in \mathbb{R}[S_1 \cap S_2]$$

$$d_1 d_2 (\tau) = \nu \, \mu \,(\tau)$$

$$= \nu \,(\tau, \tau) = \tau - \tau = 0$$

as an element of $\mathbb{R}(S_1)$ - - - of $\mathbb{R}(S_2)$

Example: Say $G_1 \cup G_2 = G = (V, E)$

$G_1 = (V_1, E_1)$, $\quad G_2 = (V_2, E_2)$

$$\begin{tikzcd}
e_0(G_1 \cap G_2) \arrow[d, equal] & e_0(G_1) \oplus e_0(G_2) \arrow[d] & & e_0(G_1 \cup G_2) \\
\mathbb{R}[V_1 \cap V_2] & \mathbb{R}[V_1] & \mathbb{R}[V_2] \arrow[ul, hook] & \mathbb{R}[V_1 \cup V_2] \arrow[u, hook]
\end{tikzcd}$$

$\mathbb{R}[V_1 \cap V_2]$ = vertex set of $G_1 \cap G_2$.

Next time we'll prove

$$0 \to \mathbb{R}[S_1 \cap S_2] \xrightarrow{\mu} \mathbb{R}[S_1] \oplus \mathbb{R}[S_2] \xrightarrow{\nu} \mathbb{R}[S_1 \cup S_2] \to 0$$

is exact.