CPSC 531F Jan 31, 2025

Mayer-Vietoris sequence:

Last time:

S_1, S_2 sets, then:

$\mathbb{R}[S] \overset{\text{def}}{=}$ formal $\mathbb{R}$-linear sums in S

$G = (V, E)$:

0-forms $\mathcal{C}_0(G) = \mathbb{R}[V]$

$= \left\{ \alpha_1 v_1 + \dots + \alpha_r v_r \;\middle|\; \begin{array}{l} v_i \in V \\ \alpha_i \in \mathbb{R} \end{array} \right\}$

$e_1(G)$ (1-forms)

$\mathbb{R}[E]$, but it's better:

$$\mathbb{R}\big[\{[v,v'] \mid \{v,v'\} \in E\}\big]$$

with $[v,v'] = -[v',v]$

There's a natural short exact sequence

$$0 \to \mathbb{R}[S_1 \cap S_2] \to (\mathbb{R}[S_1] \oplus \mathbb{R}[S_2]) \to \mathbb{R}[S_1 \cup S_2] \to 0$$

$$S_1 \cap S_2 \subset S_1$$

$$\mathbb{R}[S_1 \cap S_2] \subset \mathbb{R}[S_1]$$

e.g. $S_1 = \{A, B, C, D\}$ $\quad S_1 \cap S_2 = \{C, D\}$

$S_2 = \{C, D, E\}$

$$9C - \sqrt{2}D \in \mathbb{R}[S_1 \cap S_2]$$

in $\mathbb{R}[S_1]$ have $9C - \sqrt{2}D$

and $4A - 4A = 0$

in both $\mathbb{R}[S_1]$, $\mathbb{R}[S_1 \cap S_2]$

$$\begin{array}{ccccccccc} \sigma_3 & & d_3 & & \sigma_2 & d_2 & \sigma_1 & d_1 & \sigma_0 \end{array}$$

Trick:

$$0 \to \mathbb{R}[S_1 \cap S_2] \xrightarrow{\mu} \mathbb{R}[S_1] \oplus \mathbb{R}[S_2] \xrightarrow{\nu} \mathbb{R}[S_1 \cup S_2] \to 0 \qquad (*)$$

$$\boxed{\begin{aligned} \mu(\tau) &= (\tau, \tau) \\ \nu(\sigma_1, \sigma_2) &= \sigma_1 - \sigma_2 \end{aligned}}$$

Example:

$$9C - \sqrt{2}D \xrightarrow{\mu} (9C - \sqrt{2}D,\ 9C - \sqrt{2}D)$$

$$\xrightarrow{\nu} (9C - \sqrt{2}D) - (9C - \sqrt{2}D)$$

$$= 0$$

Claim: (*) is exact:

Exact:

$$\cdots \to \bar{U}_1 \xrightarrow{d_1} \bar{U}_0 \xrightarrow{d_0} \bar{U}_{-1} \to \cdots$$

$\mathrm{Image}(d_{i+1}) = \ker(d_i)$ in position i

$(\bar{U}_\bullet, d_\bullet)$

exact: exact in all positions

So (*) at $\bar{U}_3$

$$\begin{tikzcd} 0 \arrow[r, "d_3"] & 0 \arrow[r, "d_2"] & \mathbb{R}[S_1 \cap S_2] \\ & \bar{U}_3 \arrow[u] & \bar{U}_2 \arrow[u] \end{tikzcd}$$

$$\operatorname{image}(d_3) = \ker(d_2) = 0$$

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at $\bar{U}_2$ (or 2^{nd} position)

$$\begin{tikzcd}
\bar{U}_3 \arrow[r, "d_3"] \arrow[d] & \bar{U}_2 \arrow[r, "d_2"] \arrow[d] & \bar{U}_1 \arrow[d] \\
0 \arrow[r, "0"] & \mathbb{R}[S_1 \cap S_2] \arrow[r] & \mathbb{R}[S_1] \oplus \mathbb{R}[S_2]
\end{tikzcd}$$

$d_2 = \mu$

$\tau \longmapsto (\tau, \tau)$

$$0 \xrightarrow{d_3} \bar{U}_2 \xrightarrow{d_2} \bar{U}_1$$

exact $\iff$ $\underbrace{\operatorname{Image}(d_3)}_{0} = \ker(d_2)$

$0 = \ker(d_2)$

So

d_2 has kernel $= 0$

or $d_2 : \bar{U}_2 \to \bar{U}_1$ is an injection

Since $\mu(\tau) = (\tau, \tau)$

and $\tau \neq 0$ in $\mathbb{R}[S_1 \cap S_2]$

then $(\tau, \tau) \neq 0$ in $\mathbb{R}[S_1] \oplus \mathbb{R}[S_2]$

=

Next:

$$C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$$

$$\mathbb{R}[S_1 \cap S_2] \xrightarrow{\mu} \mathbb{R}[S_1] \oplus \mathbb{R}[S_2] \xrightarrow{\nu} \mathbb{R}[S_1 \cup S_2]$$

claim

$$\mathrm{Im}(\mu) = \ker(\nu)$$

$$\{(\tau, \tau) \mid \tau \in \mathbb{R}[S_1 \cap S_2]\}$$

$$\ker(\gamma) = \left\{ (\sigma_1, \sigma_2) \;\middle|\; \begin{array}{l} \sigma_1 \in \mathbb{R}[S_1] \\ \sigma_2 \in \mathbb{R}[S_2] \end{array} \right.$$

$$\left. \text{s.t.} \quad \sigma_1 - \sigma_2 = 0 \quad \text{in } \mathbb{R}[S_1 \cup S_2] \right\}$$

$$= \left\{ (\overset{\mathbb{R}[S_1]}{\sigma_1}, \overset{\mathbb{R}[S_2]}{\sigma_2}) \;\middle|\; \sigma_1 = \sigma_2 \text{ in } \mathbb{R}[S_1 \cup S_2] \right\}$$

Implies $\sigma_1 = \sigma_2 \in \mathbb{R}[S_1 \cap S_2]$

$=$

Example:

$S_1 = \{A, B, C, D\}$

$S_2 = \{C, D, E\}$

$S_1 \cap S_2 = \{C, D\}$

$\_\,A + \_\,B + \_\,C + \_\,D$

$= \_\,C + \_\,D + \_\,E$

0

$A + 2B \in \mathbb{R}[S_1]$

not in $\mathbb{R}[S_2]$

$(A+2B,\ \sigma_2) \xrightarrow{\gamma} A + 2B - \sigma_2 = 0$

$\sigma_2 \notin \mathbb{R}[S_2]$

Now need

$$\mathbb{R}[S_1] \oplus \mathbb{R}[S_2] \xrightarrow{\gamma} \mathbb{R}[S_1 \cup S_2] \to 0$$

$$\gamma(\sigma_1, \sigma_2) = \sigma_1 - \sigma_2$$

exact!

$$\mathrm{Im}(\gamma) = \ker\left(\mathbb{R}[S_1 \cup S_2] \to 0 \right)$$

$$= \mathbb{R}[S_1 \cup S_2]$$

i.e. γ is surjective (onto).

Example:

$$S_1 = \{A, B, C, D\}$$

$$S_2 = \{C, D, E\} \qquad S_1 \cap S_2 = \{C, D\}$$

$$S_1 \cup S_2 = \{A, B, C, D, E\}$$

typical elm of $\mathbb{R}[S_1 \cup S_2]$

$$\underbrace{\alpha_1 A + \alpha_2 B + \alpha_3 C + \alpha_4 D}_{\sigma_1} + \underbrace{\alpha_5 E}_{-\sigma_2}$$

$(\alpha_i \in \mathbb{R})$

$$\sigma_2 = -\alpha_5 E$$

$$\sigma_1 - \sigma_2 =$$

A <u>short exact</u> sequence is an exact sequence

$$0 \to U_2 \xrightarrow{d_2} U_1 \xrightarrow{d_1} U_0 \to 0$$

means: d_2 is injective (2nd pos)

d_1 is surjective (0th pos)

$\text{Image}(d_2) = \ker(d_1)$ (1st pos)

So:

$$0 \to \mathbb{R}(S_1 \cap S_2) \xrightarrow{\mu} \mathbb{R}[S_1] \oplus \mathbb{R}[S_2] \xrightarrow{\nu} \mathbb{R}(S_1 \cup S_2) \to 0$$

is a short exact sequence.

Now: we got!

$$G=(V,E), \qquad G_1=(V_1,E_1)\subset G$$

$$G_2=(V_2,E_2)\subset G$$

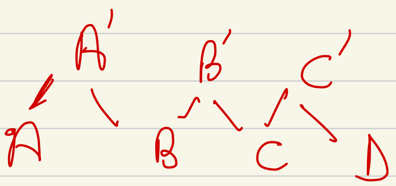
and

$$G_1\cup G_2 = G : \quad V_1\cup V_2 = V$$

$$E_1\cup E_2 = E$$

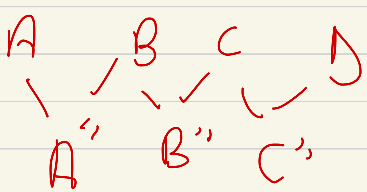
$$G_1\cap G_2 \overset{def}{=} (V_1\cap V_2,\ E_1\cap E_2)$$

G_1



to keep in mind

G_2



$$\mathcal{C}_0(G) = \mathbb{R}[V]$$

$$\mathcal{C}_0(G_1) = \mathbb{R}[V_1], \quad \mathcal{C}_0(G_2) = \mathbb{R}[V_2]$$

$$\mathcal{C}_0(G_1 \cap G_2) = \mathbb{R}[V_1 \cap V_2]$$

So

$$0 \to \mathcal{C}_0(G_1 \cap G_2) \xrightarrow{\mu_0} \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2) \xrightarrow{\nu_0} \mathcal{C}_0(G_1 \cup G_2) \to 0$$

Similarly

$$0 \to \mathcal{C}_1(G_1 \cap G_2) \xrightarrow{\mu_1} \mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2) \xrightarrow{\nu_1} \mathcal{C}_1(G_1 \cup G_2) \to 0$$

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{C}_1(G_1 \cap G_2) \arrow[r, "\mu_1"] \arrow[d, "\partial_1"'] & \mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2) \arrow[r, "\nu_1"] \arrow[d] & \mathcal{C}_1(G_1 \cup G_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{C}_0(G_1 \cap G_2) \arrow[r, "\mu_0"'] & \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2) \arrow[r, "\nu_0"'] & \mathcal{C}_0(G_1 \cup G_2) \arrow[r] & 0
\end{tikzcd}
$$

$$\partial_1 [v, v'] \overset{\text{def}}{=} [v'] - [v]$$

Claim: This diagram commutes.

$$
\begin{tikzcd}
\mathcal{C}_1(G_1 \cap G_2) \arrow[r, "\mu_1"] \arrow[d, "\partial_1"'] & \mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2) \arrow[d, "\partial_1 \quad \partial_1"'] \\
\mathcal{C}_0(G_1 \cap G_2) \arrow[r, "\mu_0"'] & \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2)
\end{tikzcd}
$$