CPSC 531F   Feb 3, 2025

Today: $G = G_1 \cup G_2$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}_1(G_1 \cap G_2) \arrow[r] \arrow[d, "\partial_1"] & \mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2) \arrow[r] \arrow[d, "\partial_1"] & \mathcal{C}_1(G) \arrow[r] \arrow[d, "\partial_1"] & 0 \\
0 \arrow[r] & \mathcal{C}_0(G_1 \cap G_2) \arrow[r] & \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2) \arrow[r] & \mathcal{C}_0(G) \arrow[r] & 0
\end{tikzcd}$$

"Short exact sequences" to "long exact"

$$\begin{tikzcd}
0 \arrow[r] & H_1(G_1 \cap G_2) \arrow[r] & H_1(G_1) \oplus H_1(G_2) \arrow[r] & H_1(G) \arrow[dll, "\delta"'] & \\
& H_0(G_1 \cap G_2) \arrow[r] & H_0(G_1) \oplus H_0(G_2) \arrow[r] & H_0(G) \arrow[r] & 0
\end{tikzcd}$$

Using: $\partial_1 : \mathcal{C}_1(G) \longrightarrow \mathcal{C}_0(G)$ $\partial_1[v,v'] = [v'] - [v]$

$$H_1(G) \overset{\text{def}}{=} \ker(\partial_1), \quad H_0(G) \overset{\text{def}}{=} \operatorname{coker}(\partial_1)$$

Admin:

(1) Hand in some of A.1 – A.9 by break week Feb 17.

(2) One we have Mayer-Vietoris, we'll have more computations to make, more exercises.

Commutative Diagram

$$
\begin{tikzcd}
\tau \arrow[rr, mapsto, "\mu_1"] \arrow[ddd, bend right] & & (\tau, \tau) \arrow[d, "\partial_1"] \\
C_1(G_1 \cap G_2) \arrow[r, "\mu_1"] \arrow[d, "\partial_1"] & \underbrace{C_1(G_1) \oplus C_1(G_2)} \arrow[d, "\partial_1"] & \partial_1(\tau, \tau) \\
C_0(G_1 \cap G_2) \arrow[r, "\mu_0"] & \overbrace{C_0(G_1) \oplus C_0(G_2)} & (\partial_1 \tau, \partial \tau) \arrow[d, <->] \\
\partial_1 \tau \arrow[rr, "\mu_0"] & & (\partial_1 \tau, \partial \tau)
\end{tikzcd}
$$

$\mu_1 : \tau \in C_1(G_1 \cap G_2)$

$\mu_1(\tau) = (\tau, \tau) \in C_1(G_1) \oplus C_2(G_2)$

2 short exact sequences, connected with a compatible "∂_1"

$$0 \to H_1(G_1 \cap G_2) \xrightarrow{\text{"}\mu_1\text{"}} H_1(G_1) \oplus H_1(G_2) \xrightarrow{\text{"}\nu_1\text{"}} H_1(G) \xrightarrow{\partial} H_0(G_1 \cap G_2) \xrightarrow{\text{"}\mu_0\text{"}} H_0(G_1) \oplus H_0(G_2) \xrightarrow{\text{"}\nu_0\text{"}} H_0(G) \to 0$$

$$\begin{array}{ccc} \tau \searrow & & (\tau \ , \ \tau) \\ H_1(G_1 \cap G_2) & \xrightarrow{\text{"}\mu_1\text{"}} & H_1(G_1) \oplus H_2(G_2) \\ \uparrow & & \\ \ker \partial_1(G_1 \cap G_2) & & \end{array}$$

$\tau \in C_1(G_1 \cap G_2)$ s.t. $\partial \tau = 0$

is $(\tau, \tau) \in C_1(G_1) \oplus C_1(G_2)$

in $\ker \partial_1$?

$$\partial_1(\tau, \tau) = (\partial_1 \tau, \partial_1 \tau)$$

$$
\begin{tikzcd}
\tau \in C_1(G_1 \cap G_2) \arrow[r] \arrow[d, "\partial_1"'] & C_1(\text{---}) \arrow[d, "\partial_1"'] \\
0 \arrow[r] & 0
\end{tikzcd}
$$

$$\mu_1 \tau \in \ker(\partial_1)$$

Gives map

$$H_1(G_1 \cap G_2) \to H_1(G_1) \oplus H_1(G_2)$$

Exercise: Show similarly

$$H_1(G_1) \oplus H_1(G_2) \xrightarrow{\text{"}\nu_1\text{"}} H_1(G)$$

is well-defined

Next: $\tau \xrightarrow{\mu_0}$

$$H_0(G_1 \cap G_2) \longrightarrow H_0(G_1) \oplus H_0(G_2)$$

then "μ_0", $\mu_0 : e_0(G_1 \cap G_2) \to$ ___

$$H_0(G_1 \cap G_2) = \operatorname{coker} \partial_1$$

$$= e_0(G_1 \cap G_2) / \operatorname{Image} e_1(G_1 \cap G_2)$$

$$\tau \in \operatorname{coker}(\partial_1)$$

$$= \{ \tau \in e_0(G_1 \cap G_2) \} / \operatorname{Image} e_1(G_1 \cap G_2)$$

$$\left.\begin{array}{l} \tau \xrightarrow{\mu_0} (\tau, \tau) \\ \hat{\tau} \xrightarrow{\mu_0} (\hat{\tau}, \hat{\tau}) \end{array}\right\} \begin{array}{l} \text{If } \tau, \hat{\tau} \text{ same} \\ \text{elemt of } H_0(G_1 \cap G_2) \end{array}$$

$$\beta \longmapsto (\beta, \beta)$$

$$\begin{array}{ccc}
\mathcal{C}_1(G_1 \cap G_2) & & \mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2) \\
\downarrow \partial_1 & & \downarrow \partial_1 \\
\mathcal{C}_0(G_1 \cap G_2) & \xrightarrow{\mu_0} & (\partial_1 \beta, \partial_1 \beta)
\end{array}$$

$\tau, \hat{\tau} \in$

$(\partial_1 \beta) = \tau - \hat{\tau}$

$\tau, \hat{\tau}$ same element of $H_0(G_1 \cap G_2)$

if $\quad \tau - \hat{\tau} \in \mathrm{Im}(\partial_1)$

$$\tau - \hat{\tau} = \partial_1 \beta$$

Claim:

$$\mu_0 \tau - \mu_0 \hat{\tau} = (\tau - \hat{\tau}, \tau - \hat{\tau})$$

$$= (\partial_1 \beta, \partial_1 \beta)$$

$$= \mu_0(\partial_1 \beta)$$

Hence

$$\mu_0 \tau - \mu_0 \hat{\tau} \in \operatorname{Im} \partial_1 \qquad \begin{array}{c} C_1(G_1) \oplus C_1(G_2) \\ \downarrow \\ C_0(G_1) \oplus C_0(G_2) \end{array}$$

So

$\mu_0 \tau, \mu_0 \hat{\tau}$ are the same element of

$$H_0(G_1) \oplus H_0(G_2)$$

$$= C_0(G_1) \oplus C_0(G_2) \Big/ \operatorname{Image}(\partial_1)$$

$$0 \to H_1(G_1 \cap G_2) \xrightarrow{\text{"}\mu_1\text{"}} H_1(G_1) \oplus H_1(G_2) \xrightarrow{\text{"}\nu_1\text{"}} H_1(G) \xrightarrow{\partial}$$

$$\xrightarrow{\partial} H_0(G_1 \cap G_2) \xrightarrow{\text{"}\mu_0\text{"}} H_0(G_1) \oplus H_0(G_2) \xrightarrow{\text{"}\nu_0\text{"}} H_0(G) \to 0$$

$$(\beta_1 \,,\, \beta_2) \xrightarrow{\nu} \beta$$

$$\begin{tikzcd}
0 \arrow[r] & C_1(G_1 \cap G_2) \arrow[r] \arrow[d, "\partial_1"'] & C_1(G_1) \oplus C_1(G_2) \arrow[r, "\nu_1"] \arrow[d, "\partial_1"'] & C_1(G) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_0(G_1 \cap G_2) \arrow[r] & C_0(G_1) \oplus C_0(G_2) \arrow[r] & C_0(G) \arrow[r] & 0
\end{tikzcd}$$

$$\alpha \longrightarrow (\partial_1 \beta_1 , \partial_1 \beta_2) \longrightarrow 0$$

Let $\beta \in H_1(G)$. By surjectivity on the top row, 2^{nd} arrow

$$\beta = \nu_1 (\beta_1 , \beta_2) = \beta_1 - \beta_2$$

$$
\begin{tikzcd}
 & \beta_1, \beta_2 \arrow[r, mapsto, "\nu_1"] \arrow[d, mapsto] & \beta_1 - \beta_2 = \beta \arrow[d, mapsto, "\partial_1"] \\
\alpha \arrow[r, mapsto, "\mu_0"] & (\partial_1\beta_1, \partial_1\beta_2) \arrow[r, mapsto, "\nu_0"] & 0
\end{tikzcd}
$$

Concretely

$$\alpha \xmapsto{\mu_0} (\alpha, \alpha)$$

$$\partial_1\beta_1 - \partial_1\beta_2 = 0$$

$$\partial_1\beta_1 = \partial_1\beta_2$$

$$\alpha \in C_0(G_1 \cap G_2)$$

$$\alpha \mapsto \alpha + \mathrm{Im}(\partial_1) \in H_0(G_1 \cap G_2)$$

We claim: the α we get doesn't depend on β_1, β_2 we chose...

$$(\hat{\beta}_1, \hat{\beta}_2) \longrightarrow \beta$$

$$(\beta_1, \beta_2) \xrightarrow{\nu} \beta$$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}_1(G_1 \cap G_2) \arrow[r] \arrow[d, "\partial_1"] & \mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2) \arrow[r, "\nu_1"] \arrow[d, "\partial_1"] & \mathcal{C}_1(G) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{C}_0(G_1 \cap G_2) \arrow[r] & \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2) \arrow[r] & \mathcal{C}_0(G) \arrow[r] & 0
\end{tikzcd}$$

$$\alpha \longrightarrow (\partial_1 \beta_1, \partial_1 \beta_2) \longrightarrow 0$$

$$\hat{\alpha} \longrightarrow (\partial_1 \hat{\beta}_1, \partial_1 \hat{\beta}_2)$$

Are $\alpha, \hat{\alpha}$ the same class in H_0?

$$(\hat{\beta}_1, \hat{\beta}_2) \xmapsto{\nu_1} \beta$$

$$(\beta_1, \beta_2) \xmapsto{\nu_1} \beta$$

$$(\beta_1, \beta_2) - (\hat{\beta}_1, \hat{\beta}_2) \xmapsto{\nu_1} 0$$

$$\tau \xrightarrow{\mu_1} \beta_1, \beta_2 - (\tilde{\beta}_1, \tilde{\beta}_2)$$

$$(\hat{\tilde{\beta}}_1, \hat{\tilde{\beta}}_2) \longrightarrow \beta$$

$$(\beta_1, \beta_2) \xrightarrow{\nu} \beta$$

$$\begin{tikzcd}
0 \arrow[r] & C_1(G_1 \cap G_2) \arrow[r] \arrow[d, "\partial_1"'] & C_1(G_1) \oplus C_1(G_2) \arrow[r, "\nu_1"] \arrow[d, "\partial_1"] & C_1(G) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_0(G_1 \cap G_2) \arrow[r] & C_0(G_1) \oplus C_0(G_2) \arrow[r] & C_0(G) \arrow[r] & 0
\end{tikzcd}$$

$$\tau \mapsto \partial_1 \tau$$

$$\alpha \longrightarrow (\partial_1 \beta_1, \partial_1 \beta_2) \longrightarrow 0$$

$$\partial_1 \tau \begin{cases} \alpha \longrightarrow (\partial_1 \beta_1, \partial_1 \beta_2) \\ \hat{\tilde{\alpha}} \longrightarrow (\partial_1 \hat{\beta}_1, \partial_1 \hat{\beta}_2) \end{cases}$$

δ_0

$$\tau \longrightarrow (\beta_1, \beta_2) - (\tilde{\beta}_1, \tilde{\beta}_2) - 0$$

$$\partial_1 \tau \longrightarrow \alpha - \tilde{\alpha}$$

$$\alpha - \tilde{\alpha} \in \operatorname{Im}\big(\partial_1 (G_1 \cap G_2)\big)$$

Hence!

$\alpha, \tilde{\alpha}$ as lying in $H_0(G_1 \cap G_2)$

they represent the same class!

Hence

$$H_1(G) \xrightarrow{\delta} H_0(G_1 \cap G_2)$$

$$\beta \overset{\text{make choices}}{\rightsquigarrow} \alpha$$

but α we get doesn't depend on choices!