CPSC 531F Feb 5, 2025

Admin:

- Monday's class missed...
- Assume no Monday (Feb 3) class

Break: Feb 17-21

Turn in, say, some Exercises A.1–A.9

sometime during Feb 17-21 TO

GET SOME FEEDBACK.

email: jf@cs.ubc.ca

Subject: CPSC 531F (somewhere)

"Real due date" last day of term.

Today: $G=(V,E)$, $G_1=(V_1,E_1)$, $G_2=(V_2,E_2)$

Last time: $G_1, G_2 \subseteq G$ graphs, $G_1 \cup G_2 = G$

$$0 \to C_0(G_1 \cap G_2) \xrightarrow{\mu_0} C_0(G_1) \oplus C_0(G_2) \xrightarrow{\nu_0} C_0(G_1 \cup G_2) \to 0$$

short exact, based on

$$0 \to \mathbb{R}[V_1 \cap V_2] \xrightarrow{\mu_0} \mathbb{R}[V_1] \oplus \mathbb{R}[V_2] \xrightarrow{\nu_0} \mathbb{R}[V_1 \cup V_2] \to 0$$

injective   surjective

we took

$\mu_0(\tau) \mapsto (\tau, \tau)$

$\nu_0(\sigma_1, \sigma_2) \mapsto \sigma_1 - \sigma_2$

μ_0 is injective ($\tau \neq 0$, then $(\tau, \tau) \neq (0,0)$)

$\Leftrightarrow$ exactness at $\mathbb{R}[V_1 \cap V_2]$.

ν_0 is sujective $\Longleftrightarrow$ exactness at $\mathbb{R}[V_1 \cup V_2]$

$\mathrm{Im}(\mu_0) = \mathrm{Ker}(\nu_0) \Leftrightarrow \{(\tau,\tau)\} = \ker(\nu_0) \Leftrightarrow$ exactness in middle

S is a set, $\mathbb{R}[S] = \{$ formal $\mathbb{R}$-linear combinations of elements of $S \}$

Similarly

$$0 \to \mathcal{C}_1(G_1 \cap G_2) \xrightarrow{\mu_1} \mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2) \xrightarrow{\nu_1} \mathcal{C}_1(G_1 \cup G_2) \to 0$$

$$\mathcal{C}_1(G) \overset{\text{def}}{=} \left\{ \begin{array}{l} \text{formal } \mathbb{R}\text{-linear sums } [v,v'] \\ \text{s.t. } v, v' \in \text{edge of } G \end{array} \right\}$$

$$[v,v'] = -[v',v]$$

Hence

$$\left.\begin{tikzcd}
0 \arrow[r] \arrow[d, dotted] & \mathcal{C}_1(G_1 \cap G_2) \arrow[r, "\mu_1"] \arrow[d, "\partial_1"'] & \mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2) \arrow[r, "\nu_1"] \arrow[d, "\partial_1"'] & \mathcal{C}_1(G_1 \cup G_2) \arrow[r] \arrow[d, "\partial_1"'] & 0 \arrow[d, dotted] \\
0 \arrow[r] & \mathcal{C}_0(G_1 \cap G_2) \arrow[r, "\mu_0"] & \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2) \arrow[r, "\nu_0"] & \mathcal{C}_0(G_1 \cup G_2) \arrow[r] & 0
\end{tikzcd}\right\} (*)$$

$$\partial_1 [v,v'] = [v'] - [v]$$

Claim: (*) is commutative:

$$[v,v'] \qquad\qquad \mu_1[v,v'] = ([v,v'],[v,v'])$$

$$\begin{tikzcd}
C_1(G_1 \cap G_2) \arrow[r, "\mu_1"] \arrow[d, "\partial_1"'] & C_1(G_1) \oplus C_1(G_2) \arrow[d, "\partial_1"] \\
C_0(G_1 \cap G_2) \arrow[r, "\mu_0"] & C_0(G_1) \oplus C_0(G_2)
\end{tikzcd}$$

$$[v'] - [v] \qquad\qquad (\partial_1[v,v'], \partial_1[v,v'])$$

$$\mu_0 \qquad\qquad \|$$

$$([v']-[v], [v']-[v])$$

Same

This allows for "diagram chasing"

Similarly 2^{nd} part of (*) is commutative, and everything else

Claim: (*)

$$\begin{tikzcd} 0 \arrow[r] & C_1(G_1 \cap G_2) \arrow[r, "\mu_1"] \arrow[d, "\partial_1"] & C_1(G_1) \oplus C_1(G_2) \arrow[r, "\nu_1"] \arrow[d, "\partial_1"] & C_1(G_1 \cup G_2) \arrow[r] \arrow[d, "\partial_1"] & 0 \\ 0 \arrow[r] & C_0(G_1 \cap G_2) \arrow[r, "\mu_0"] & C_0(G_1) \oplus C_0(G_2) \arrow[r, "\nu_0"] & C_0(G_1 \cup G_2) \arrow[r] & 0 \end{tikzcd} \quad (*)$$

Gives us a map

$$\begin{tikzcd} 0 \arrow[r] & H_1(G_1 \cap G_2) \arrow[r, "``\mu_1"'] & H_1(G_1) \oplus H_1(G_2) \arrow[r, "``\nu_1"'] & H_1(G_1 \cup G_2) \arrow[dlll, "\delta"] & \\ & H_0(G_1 \cap G_2) \arrow[r, "``\mu_0"'] & H_0(G_1) \oplus H_0(G_2) \arrow[r, "\nu_0"'] & H_0(G_1 \cup G_2) \arrow[r] & 0 \end{tikzcd} \quad (**)$$

which is exact.

δ above is the truly remarkable. The other maps are pretty direct

By def:

$$\partial_1 : \mathcal{C}_1(G) \longrightarrow \mathcal{C}_0(G)$$

$$[v, v'] \longmapsto [v'] - [v]$$

$$H_1(G) = \ker(\partial_1)$$

$$H_0(G) = \operatorname{coker}(\partial_1) = \mathcal{C}_0(G) / \operatorname{Image}(\partial_1)$$

$$H_1(G_1 \cap G_2) \overset{def}{=} \ker \partial_1 : \mathcal{C}_1(G_1 \cap G_2) \longrightarrow \mathcal{C}_0(G_1 \cap G_2)$$

$$\tau \in H_1(G_1 \cap G_2)$$

$$\Longleftrightarrow \partial_1 \tau = 0, \quad \tau \in \mathcal{C}_1(G_1 \cap G_2)$$

$\tau \in H_1(G_1 \cap G_2) \implies$

$$\tau \xrightarrow{\mu_1} (\tau, \tau)$$

$$\begin{array}{ccccccccc}
0 \to & \mathcal{C}_1(G_1 \cap G_2) & \xrightarrow{\mu_1} & \mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2) & \xrightarrow{\nu_1} & \mathcal{C}_1(G_1 \cup G_2) & \to & 0 \\
& \downarrow \partial_1 & & \downarrow \partial_1 & & \downarrow \partial_1 & & \\
0 \to & \mathcal{C}_0(G_1 \cap G_2) & \xrightarrow{\mu_0} & \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2) & \xrightarrow{\nu_0} & \mathcal{C}_0(G_1 \cup G_2) & \to & 0
\end{array} \quad (*)$$

$\tau \mapsto 0$

$$(\partial_1 \tau, \partial_1 \tau)$$

$$= (0, 0)$$

$$\begin{array}{ccccc}
\tau & \xmapsto{\mu_1} & ?? & \mu_1(\tau) & \mu_1(\tau) \\
\downarrow & & \downarrow \text{ whatever} & & \downarrow \partial_1 \\
0 & \longmapsto & 0 & & 0
\end{array}$$

$\implies \mu_1(\tau) \in \ker \partial_1$

Gives a map $H_1(G_1 \cap G_2) \to H_1(G_1) \oplus H_1(G_2)$

Claim: $C_0(G_1 \cap G_2) \longrightarrow C_0(G_1) \oplus C_0(G_2)$

gives a map

$$H_0(G_1 \cap G_2) \longrightarrow H_0(G_1) \oplus H_0(G_2)$$

$$\begin{matrix} \| \\ C_0(G_1 \cap G_2) \big/ \mathrm{Im}(\partial_1) \end{matrix}$$

$\tau \in H_0(G_1 \cap G_2)$

so τ is really $\begin{pmatrix} \text{elt of} \\ C_0(G) \end{pmatrix} + \underbrace{\mathrm{Im}(\partial_1)(G_1 \cap G_2)}_{I}$

$$\hat{\tau} + I$$

$\mu_0(\hat{\tau} + I) \in H_0(G_1) \oplus H_0(G_2)$

So, say $\hat{\tau}, \hat{\hat{\tau}}$ are in $C_0(G)$

but $\hat{\tau} \equiv \hat{\hat{\tau}} \pmod{I}$

$\mathrm{Image}(\partial_1 |_{G_1 \cap G_2})$

$$\mu_0(\hat{\tau}) = (\hat{\tau}, \hat{\tau})$$

$$\mu_0(\hat{\hat{\tau}}) = (\hat{\hat{\tau}}, \hat{\hat{\tau}})$$

but are

$(\hat{\tau}, \hat{\tau})$ and $(\hat{\hat{\tau}}, \hat{\hat{\tau}})$

these the same coset in

$$H_0(G_1) \oplus H_0(G_2) = C_0(G_1) / \mathrm{Im}\, \partial_1(G_1) \oplus C_0(G_2) / \mathrm{Im}\, \partial_1(G_2)$$

Subtlety: If

$$\left.\begin{array}{l} U_1 \subset U \\ W_1 \subset W \end{array}\right\} \mathbb{R}\text{-vector spaces}$$

and

$$\mu : U \to W$$

you only get a map

$$U/U_1 \longrightarrow W/W_1$$

well-defined if

$$\mu(U_1) \subset W_1$$

Example

$$\mathbb{Z}/3\mathbb{Z} = \left\{ \begin{array}{l} 0 + 3\mathbb{Z}, \\ 1 + 3\mathbb{Z}, \\ 2 + 3\mathbb{Z} \end{array} \right\}$$

$$1 + \underbrace{3\mathbb{Z}} = \{ -5, -2, 1, 4, 7, 10, \dots \}$$

$$1 + \{ \dots -3, 0, 3, 6, \dots \}$$

Say $\underset{3\mathbb{Z}}{\bar{U}_1} \in \underset{\mathbb{Z}}{\bar{U}}$

$$\bar{U} / \bar{U}_1 = \mathbb{Z}/3\mathbb{Z} =$$

$W_1 = 0, \quad W = \mathbb{Z}$

$$U \xrightarrow{id} W$$

$a \in \mathbb{Z} \longmapsto a \in \mathbb{Z}$ identity

$U/U_1 \overset{?}{\rightsquigarrow} W/W_1$??

$$\begin{array}{ccc} a & \longmapsto & a \\ \mathbb{Z} = U & \longrightarrow & W = \mathbb{Z} \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & 2 \\ 4 & \longmapsto & 4 \\ & \vdots & \end{array}$$

$U/U_1 = \mathbb{Z}/3\mathbb{Z} \overset{?}{\rightsquigarrow} W/W_1 = \mathbb{Z}$??

But

$$\mathbb{Z}/6\mathbb{Z} \longrightarrow \mathbb{Z}/3\mathbb{Z}$$

$$a \in \mathbb{Z} \longrightarrow a \in \mathbb{Z}$$

$$6\mathbb{Z}$$

$$\{\ldots -6, 0, 6, \ldots\} \longrightarrow \{\ldots, -3, 0, 3, 6, \ldots\}$$