CPSC 531F Feb 7, 2025

$$(AB)^{-1} = B^{-1}A^{-1}$$

Conceptually: put on socks, then shoes

inverse take off shoes, then socks..

Birthday Jan 1 $\xrightarrow{\text{2 days later}}$ Jan 3

Feb 1 $\longrightarrow$ Feb 3

Feb 7 $\longrightarrow$ Feb 9

we don't the year ...

but -- Feb 28 $\longrightarrow$ $\begin{cases} \text{March 1} \\ \text{March 2} \end{cases}$

If today is Feb 7

this month is Feb

but if this month is feb...

day is Feb 1, Feb 2, ...

There's a map

$$\mathbb{Z}/6\mathbb{Z} \longrightarrow \mathbb{Z}/3\mathbb{Z}$$

Computer science:

"mod 6" $\qquad \mathrm{mod}(n,6) \in \{0,1,\dots,5\}$

Math: $\mathbb{Z}/6\mathbb{Z}$ "integers mod 6"

(1) $n \equiv n' \pmod 6 \iff n - n' \in 6\mathbb{Z}$

$6\mathbb{Z} = \{\dots, -6, 0, 6, 12, \dots\}$

If $n \equiv n' \mod 6$

$n' \equiv n'' \mod 6$

$\Rightarrow n \equiv n'' \mod 6$

$n \equiv n' \mod 6, \quad m \equiv m' \pmod 6$

$\Rightarrow \quad n+m \equiv n'+m' \pmod 6$

gives "+"

$\mathbb{Z}/6\mathbb{Z}$ cosets: $\{0, 1, \ldots, 5\} \hookrightarrow$

$0 + 6\mathbb{Z} = \{ \ldots, -6, 0, 6, 12, \ldots \}$

$1 + 6\mathbb{Z} = \{ \ldots, -5, 1, 7, 13, \ldots \}$

$2 + 6\mathbb{Z}$

$\vdots$

$5 + 6\mathbb{Z}$

$n \equiv n' \pmod 6$ $\quad$ $m \equiv m' \mod 6$

$n \cdot m \equiv n' \cdot m' \mod 6$

$(n + 6\mathbb{Z}) \cdot (m + 6\mathbb{Z})$

$\in (n+m) + (6\mathbb{Z})$

so $\cdot$ defined mod 6, $\mathbb{Z}/6\mathbb{Z}$

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n^m $\sim$?

$n \equiv n' \pmod 6$

$m \equiv m' \pmod 6$

$n^m \equiv n'^{m'} \pmod 6$???

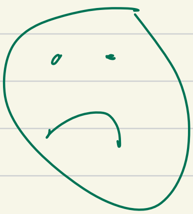
$$(n+6\mathbb{Z})^{(m+6\mathbb{Z})} = \text{unique} + 6\mathbb{Z}$$

$n = n' = 2$

$m = 0 \qquad m' = 6$

$$2^0 = 1, \qquad 2^6 = (2\cdot 2)^3 = 64$$

$$\equiv 4 \pmod{6}$$



Claim: If $f : \mathbb{Z} \to \mathbb{Z}$

given by $f(n) = n$, $id_{\mathbb{Z}}$

$$\mathbb{Z}/6\mathbb{Z} \longrightarrow \mathbb{Z}/3\mathbb{Z}$$

$\mathbb{Z}/6\mathbb{Z} \ni 0 \longrightarrow 0 \in \mathbb{Z}/3\mathbb{Z}$

$1 \to 1$

$2 \to 2$

$3 \to 0$

$4 \to 1$

$5 \to 2$

[Don't get $\mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ in this way...]

$f(n) = n$

$$\mathbb{Z} \xrightarrow{f} \mathbb{Z}$$

$$\mathbb{Z}/6\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z}$$

$$6\mathbb{Z} \leftrightarrow \mathbb{Z}$$

$0 \rightarrow 0$

$6 \rightarrow 6$

$12 \rightarrow 12$

$\twoheadrightarrow$ surjection

$\hookrightarrow$ included in, subset of

can we build this

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
6\mathbb{Z} \arrow[d, hook] \arrow[r, dashed, "?"] & 3\mathbb{Z} \arrow[d, hook] \\
\mathbb{Z} \arrow[d, two heads] \arrow[r, "f"] & \mathbb{Z} \arrow[d, two heads] \\
\mathbb{Z}/6\mathbb{Z} \arrow[d] \arrow[r, dashed, "?"] & \mathbb{Z}/3\mathbb{Z} \arrow[d] \\
0 & 0
\end{tikzcd}$$

exact

$f(n) = n$

restriction $id_{\mathbb{Z}}$ to $id_{6\mathbb{Z}}$

$$\begin{tikzcd}
6 \in 6\mathbb{Z} \arrow[r, dashed] \arrow[d, hook] & 3\mathbb{Z} \ni 6 \arrow[d, hook] \\
\mathbb{Z} \arrow[r, "id"] & \mathbb{Z}
\end{tikzcd}$$

$6 \longrightarrow 6$ ☺

Now

$$\begin{tikzcd}
6\mathbb{Z} \arrow[r, dashed] \arrow[d, hook] & 3\mathbb{Z} \arrow[d, hook] \\
\mathbb{Z} \arrow[r, "id"] \arrow[d, two heads] & \mathbb{Z} \arrow[d, two heads] \\
\mathbb{Z}/6\mathbb{Z} \arrow[r, dashed] & \mathbb{Z}/3\mathbb{Z}
\end{tikzcd}$$

$5, 11, 65, -1 \in \mathbb{Z}$

$\longrightarrow 5, 11, 65$

one class??

pick $5, 11, 65, -1$ all in $5 + \mathbb{Z}/6\mathbb{Z}$

$\mathbb{Z}/6\mathbb{Z} = \{ 0 + 6\mathbb{Z},\ 1 + 6\mathbb{Z}, \dots \}$

$$
\begin{tikzcd}
(\beta-\beta')\in & 6\mathbb{Z} \arrow[r] \arrow[d, hook] & 3\mathbb{Z} \arrow[d, hook] & \beta-\beta'\in 3\mathbb{Z} \arrow[d] \\
\text{pick } \beta, \beta' \arrow[u, dashed] \arrow[d] \arrow[d, shift left=2] & \mathbb{Z} \arrow[r, "\mathcal{L}"] \arrow[d, two heads] & \mathbb{Z} \arrow[d] & \beta-\beta' \arrow[d] \\
\beta-\beta'\in 6\mathbb{Z} \quad 0\in & \mathbb{Z}/6\mathbb{Z} \arrow[r] & \mathbb{Z}/3\mathbb{Z} & 0
\end{tikzcd}
$$

If ①, then we have

Generally

$$\begin{tikzcd}
\beta-\beta' \in \bar{U}_1 \arrow[r, dashed, "\mathcal{L}|_{\bar{U}_1}"] \arrow[d, hook] & W_1 \arrow[d, hook] & \mathcal{L}(\beta)-\mathcal{L}(\beta') \arrow[d] \\
\beta,\beta' \quad \beta-\beta' \quad U \arrow[r, "\mathcal{L}"] \arrow[d] & W \arrow[d] & \mathcal{L}(\beta)-\mathcal{L}(\beta') \arrow[d] \\
\text{Same} \quad 0 \quad \bar{U}/\bar{U}_1 \arrow[r, dashed, "②"] & W/W_1 & 0
\end{tikzcd}$$

So need for ①

$$\bar{U}_1 \xrightarrow{\mathcal{L} \text{ restrict to } U_1} \text{get something in } W_1$$

Is there a map

$$\begin{tikzcd}
3\mathbb{Z} \arrow[r, "f = \text{identity restrict to } 3\mathbb{Z}"] \arrow[d, hook] & ??\ 6\mathbb{Z} \\
\mathbb{Z} \arrow[r, "n \mapsto n", "f"'] \arrow[d] & \mathbb{Z} \arrow[d] \\
(\mathbb{Z}/3\mathbb{Z}) \arrow[r, dashed, "?"] & (\mathbb{Z}/6\mathbb{Z})
\end{tikzcd}$$

$$f(2 + 3\mathbb{Z}) = (2 + \mathbb{Z}/6\mathbb{Z}) \cup (5 + \mathbb{Z}/6\mathbb{Z})$$

but not a single class

$\{-3, 0, 3, 6, \dots\}$

$$\begin{tikzcd}
3\mathbb{Z} \arrow[r, "n \mapsto 2n"] \arrow[d] & \{\dots, -6, 0, 6, 12, \dots\} \subset 6\mathbb{Z} \arrow[d, dashed, hook] \\
\mathbb{Z} \arrow[r, "n \mapsto 2n"] \arrow[d] & \mathbb{Z} \arrow[d] \\
(\mathbb{Z}/3\mathbb{Z}) \arrow[r, "!!!"] & (\mathbb{Z}/6\mathbb{Z})
\end{tikzcd}$$

$n = 0, 3, 6, \dots$ $\qquad$ $2n = 0, 6, 12$

$n = 1, 4, 7, \dots$ $\qquad$ $2n = 2, 8, 14, \dots$

Upshot: $\mathcal{L} : U \to W$

and $U_1 \subset U$ ($\mathbb{R}$-vector spaces)

$W_1 \subset W$

$\mathcal{L}$ extends to $\mathcal{L}(U_1) \subset W_1$

$$\begin{tikzcd}
U_1 \arrow[r] \arrow[d, dashed] & W_1 \arrow[d, dashed] \\
U \arrow[r, "\mathcal{L}"] \arrow[d, dashed] & W \arrow[d, dashed] \\
U/U_1 \arrow[r] & W/W_1
\end{tikzcd}$$

iff $\mathcal{L}(U_1) \subset W_1$

Monday:

$$H_0(G_1 \cap G_2) \to H_0(G_1) \oplus H_0(G_2)$$

quotient spaces