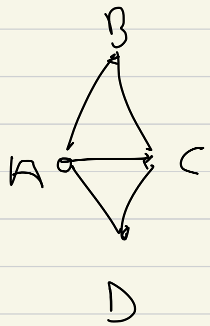
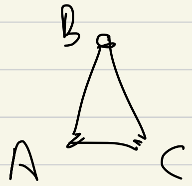
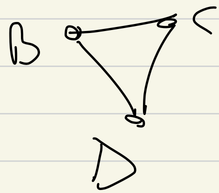
CPSC 531F Feb 10, 2025

G :



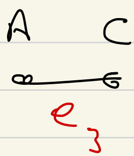


G_1



G_2

$G_1 \cap G_2$

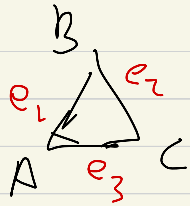


$$H_0(G_1 \cap G_2) = \frac{\mathbb{R}[A, C]}{\mathrm{Image}(\partial_1(G_1 \cap G_2))}$$

$$G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$$

$$\partial_1 [A, C) = [C] - [A)$$

G_1



$G_1 = (V_1, E_1)$

$$H_0(G_1) = \mathbb{R}[A,B,C] \Big/ \mathrm{Image}(\partial_1(G_1))$$

$$= \mathbb{R}[A,B,C] \Big/ \left([B]-[A],\ [C]-[A],\ [C]-[B] \right)$$

$e_1 : [A,B], \quad \partial_1[A,B] = [B]-[A]$

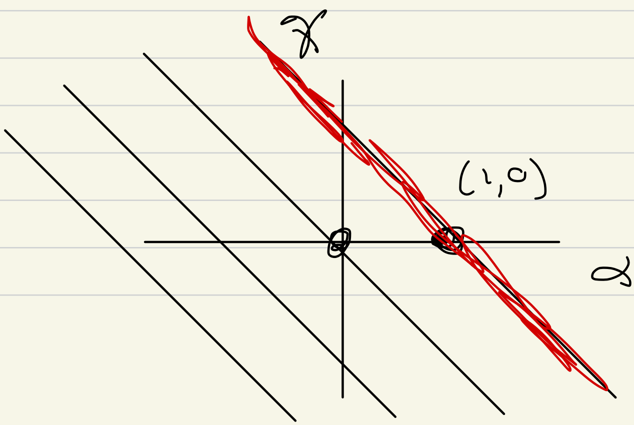
$[B,A], \quad \partial_1[B,A] = [A]-[B]$

$$\{\alpha A + \beta B + \gamma C \mid \alpha, \beta, \gamma \in \mathbb{R}\} \Big/ \begin{matrix} (-1)A + 1\cdot(B), \\ (-1)A + 1\cdot(C) \end{matrix}$$

$$\left\{ \alpha A + \beta B + \gamma C \right\} \Big/ \left\{ \tilde{\alpha} A + \tilde{\beta} B + \tilde{\gamma} C \;\middle|\; \tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} = 0 \right\}$$

$$\cong \mathbb{R}^3 \Big/ \left\{ (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) \;\middle|\; \tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} = 0 \right\}$$

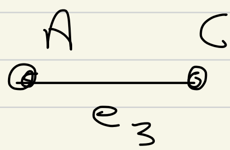
$$H_0(G_1 \cap G_2) = \left\{ \alpha A + \gamma C \right\} \Big/ \mathbb{R}(-A + C)$$



$H_0(G_1 \cap G_2)$

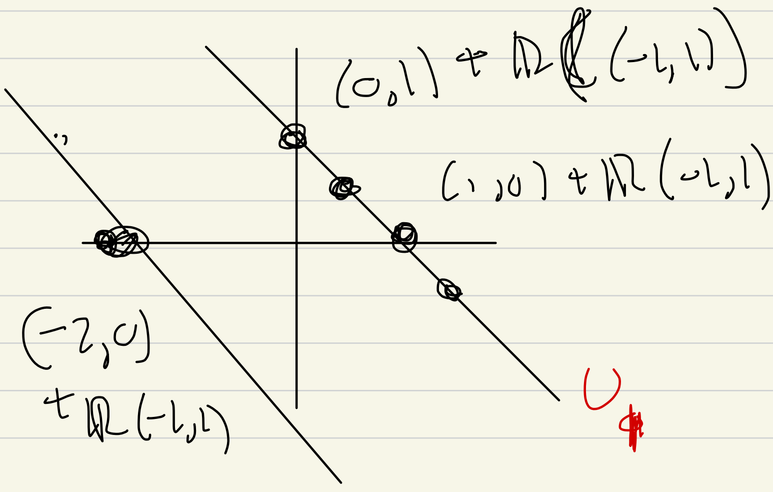
$[A] + U_1$

C

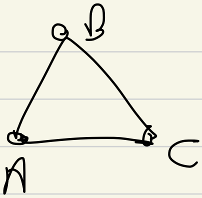
$H_0\Big($  $\Big)$

$$= \{(\alpha, \gamma) \mid \alpha, \gamma \in \mathbb{R}\} \Big/ \{(-\beta, \beta)\}$$

$$\mathrm{Span}\big((1,0) + \mathbb{R}(-1,1)\big)$$



$$H_0(G_1) = H_0\left(\quad \right)$$



is spanned by $[B] + W_1$

$$\{(\alpha,\beta,\gamma) \mid \alpha+\beta+\gamma=0\} = W_1$$

$$W/W_1$$

$$H_0(G_1 \cap G_2) \longrightarrow H_0(G_1) \qquad [B] + W_1$$

$$[A] + U_1 \longmapsto [A] + U_1 + W_1$$

$$\begin{tikzcd} U_1 \arrow[r, dashed, "\mathcal{L}_1"] \arrow[d] & W_1 \arrow[d] \\ U \arrow[r, "\mathcal{L}"] & W \end{tikzcd}$$

$$\mathbb{R}[A, E] \xrightarrow{\text{inclusion}} \mathbb{R}(A, B, C)$$

$$e_G(G_1 \cap G_2) \to e_G(G_1)$$

$$U_1 = \operatorname{Image}\left(\partial_1(G_1 \cap G_2)\right)$$

$$= \mathbb{R}\left([A] - [G]\right)$$

$$W_1 = \operatorname{Image}\left(\partial_1(G_1)\right)$$

$$= \mathbb{R}\left([B] - [A]\right) + \mathbb{R}\left([C] - [A]\right)$$

$$
\begin{tikzcd}
\mathbb{R}([C]-[A]) \arrow[r, dashed, "\text{inclusion, restricts}"] \arrow[d, hook] & \mathbb{R}([B]-[A], [C]-[A]) \arrow[d, hook] \\
\mathbb{R}([A],[C]) \arrow[r, "\text{inclusion}"] \arrow[d, "\text{quotient}"] & \mathbb{R}([A],[B],[C]) \arrow[d, "\text{quotient}"] \\
H_0(G_1 \cap G_2) \arrow[r, dashed] & H_0(G_1)
\end{tikzcd}
$$

$$H_0(G_1 \cap G_2) = \mathbb{R}\{[A],[C]\} \Big/ \mathbb{R}([C]-[A])$$

$$H_0(G_1) = \mathbb{R}([A],[B],[C]) \Big/ \mathbb{R}\{[B]-[A],\ [C]-[A]\}$$

Abstractly: say $U_1 \subset U$, $W_1 \subset W$

and

$$\begin{tikzcd} U_1 \arrow[r, "\mathcal{L}'"] \arrow[d, hook] & W_1 \arrow[d, hook] \\ U \arrow[r, "\mathcal{L}"] \arrow[d] & W \arrow[d] \\ U/U_1 \arrow[r, dashed] & W/W_1 \end{tikzcd}$$

$\mathcal{L}' := \mathcal{L}\big|_{U_1}$

then

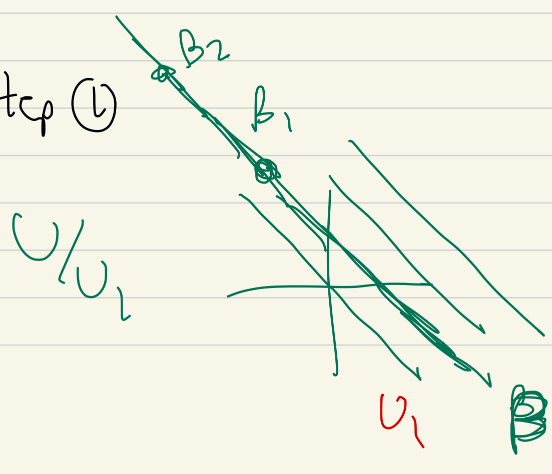
build.

Diagram chasing:

$$\begin{tikzcd}
 & U_1 \arrow[r, "\mathcal{L}'"] \arrow[d] & W_1 \arrow[d] & \\
\text{pick } \beta_1 \arrow[d] \arrow[rrr, bend left, shift left=8] & U \arrow[r, "\mathcal{L}"] \arrow[d] & W \arrow[d] & \mathcal{L}(\beta_1) = \alpha_1 \arrow[d, mapsto] \\
\beta \arrow[rr, dashed, bend right] & U/U_1 \arrow[r, dashed] & W/W_1 & \alpha_1 + W_1 \\
 & & ? &
\end{tikzcd}$$

(let's build)

Step ①



$$L(\beta_1)=\alpha_1,\ L(\beta_2)=\alpha_2$$

$$
\begin{tikzcd}
 & \beta_1-\beta_2 \in \arrow[d, mapsto] & W_1 \arrow[r, "L'"] \arrow[d] & W_1 \arrow[d] & L'(\beta_1-\beta_2) \arrow[d, mapsto] \\
\beta_2,\ \beta_1 \arrow[d, mapsto] & \beta_1-\beta_2 \arrow[d, mapsto] & U \arrow[r, "L"] \arrow[d] & W \arrow[d] & L(\beta_1-\beta_2) \arrow[d, mapsto] \\
\beta & 0 & U/U_1 & W/W_1 & 0
\end{tikzcd}
$$

So $\left.\begin{array}{l} L(\beta_1)=\alpha_1 \\ L(\beta_2)=\alpha_2 \end{array}\right\}$ map to the same element of W/W_1

Same argument:

$$
\begin{tikzcd}
U_1 \arrow[r, "\mathcal{L}'"] \arrow[d, "d_U"'] & W_1 \arrow[d, "d_W"] \\
U \arrow[r, "\mathcal{L}"] \arrow[d] & W \arrow[d] \\
U/\mathrm{Image}(d_U) \arrow[r, dashed] & W/\mathrm{Image}(d_W)
\end{tikzcd}
$$

$\mathrm{coker}(d_W)$

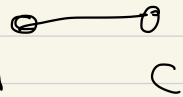
Claim with $d_W \mathcal{L}' = \mathcal{L} d_U$

$U/\mathrm{Image}(d_U) = U/d_U(U_1)$

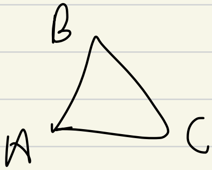
$\mathrm{coker}(d_U)$

this map is unique if you have a commutative diagram

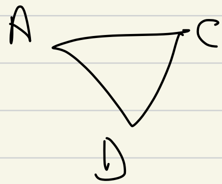
$G_1 \cap G_2$



G_1

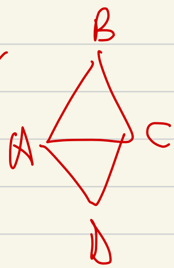


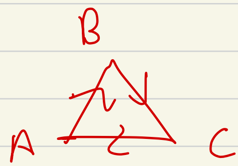
G_2



$$H_1(G_1 \cap G_2) \xrightarrow{\text{"}\mu_1\text{"}} H_1(G_1) \oplus H_1(G_2) \xrightarrow{\text{"}\nu_1\text{"}} H_1(G_1 \cup G_2)$$

$$\xrightarrow{\delta} H_0(G_1 \cap G_2) \xrightarrow{\text{"}\mu_0\text{"}} H_0(G_1) \oplus H_0(G_2) \xrightarrow{\text{"}\nu_0\text{"}} H_0(G_1 \cup G_2)$$

$H_1\Big($  $\Big)$

take 

$\partial_1 \big(\quad \big) = 0$

$$\begin{tikzcd} e_1(G_1 \cup G_2) \arrow[d] \\ e_\sigma(G_1 \cup G_2) \end{tikzcd}$$