CPSC 531F $\qquad$ Feb 12, 2025

Today: Build δ of Mayer-Vietoris!

$$\begin{tikzcd}
0 \arrow[r] & H_1(G_1 \cap G_2) \arrow[r] & H_1(G_1) \oplus H_1(G_2) \arrow[r] & H_1(G) \arrow[dlll, out=0, in=180, "\delta"'] \\
& H_0(G_1 \cap G_2) \arrow[r] & H_0(G_1) \oplus H_0(G_2) \arrow[r] & H_0(G) \arrow[r] & 0
\end{tikzcd}$$

From:

$$\begin{tikzcd}
0 \arrow[r] & C_1(G_1 \cap G_2) \arrow[r] \arrow[d, "\partial_1"'] & C_1(G_1) \oplus C_1(G_2) \arrow[r] \arrow[d, "\partial_1 \oplus \partial_1"'] & C_1(G) \arrow[r] \arrow[d, "\partial_1"'] & 0 \\
0 \arrow[r] & C_0(G_1 \cap G_2) \arrow[r] & C_0(G_1) \oplus C_0(G_2) \arrow[r] & C_0(G) \arrow[r] & 0
\end{tikzcd}$$

$$\partial_1 [v, v'] = [v'] - [v]$$

These ideas really work in a general context

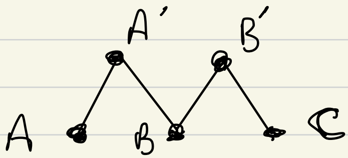
$$\begin{tikzcd}
 & \vdots & \vdots & & & \\
0 \arrow[r] & A_2 \arrow[r] \arrow[d, "\partial_{2,A}"'] & B_2 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & 0 & \text{short exact} \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d, "\partial_{1,A}"'] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 & \text{short exact} \\
0 \arrow[r] & A_0 \arrow[r] & B_0 \arrow[r] & C_0 \arrow[r] & 0 & \\
 & \vdots & \vdots & \vdots & &
\end{tikzcd}$$

chain maps $\downarrow$

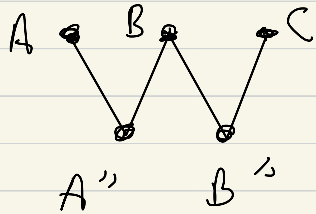
$\partial_i \partial_{i+1} = 0$

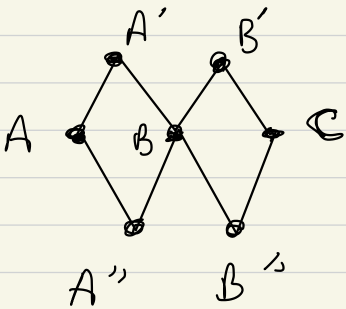
Example:

G_1

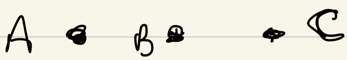


G_2



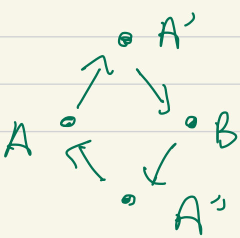


$G = G_1 \cup G_2$



$G_1 \cap G_2$

$$H_1(G) \xrightarrow{\delta} H_0(G_1 \cap G_2)$$



$\longmapsto$??

Want $H_1(G) \xrightarrow{\partial} H_0(G_1 \cap G_2)$

take $\beta \in H_1(G) = \ker \partial_1$

$$\begin{tikzcd}
 & & & \beta \arrow[d, maps to, "\partial"] \\
0 \arrow[r] & C_1(G_1 \cap G_2) \arrow[r] \arrow[d, "\partial_1"] & C_1(G_1) \oplus C_1(G_2) \arrow[r] \arrow[d, "\partial_1 \oplus \partial_1"] & C_1(G) \arrow[r] \arrow[d, "\partial_1"] & 0 \\
0 \arrow[r] & C_0(G_1 \cap G_2) \arrow[r] & C_0(G_1) \oplus C_0(G_2) \arrow[r] & C_0(G) \arrow[r] & 0 \\
 & & & 0
\end{tikzcd}$$

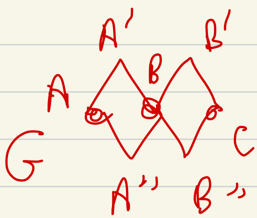
Find (σ_1, σ_2), pick one ...

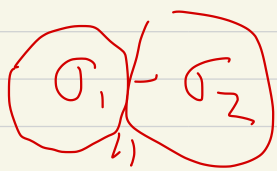
$$\begin{tikzcd}
 & & (\sigma_1, \sigma_2) \arrow[r, maps to] \arrow[d, maps to] & \beta \arrow[d, maps to, "\partial"] \\
0 \arrow[r] & C_1(G_1 \cap G_2) \arrow[r, "\mu_1"] \arrow[d, "\partial_1"] & C_1(G_1) \oplus C_1(G_2) \arrow[r, "\nu_1"] \arrow[d, "\partial_1 \oplus \partial_1"] & C_1(G) \arrow[r] \arrow[d, "\partial_1"] & 0 \\
0 \arrow[r] & C_0(G_1 \cap G_2) \arrow[r, "\mu_0"] & C_0(G_1) \oplus C_0(G_2) \arrow[r, "\nu_0"] & C_0(G) \arrow[r] & 0 \\
 & \tau \arrow[r, dashed, maps to] & (\partial_1 \sigma_1, \partial \sigma_2) \arrow[r, maps to] & 0
\end{tikzcd}$$

Example:

$$H_1(G) \xrightarrow{\delta} H_0(G_1 \cap G_2)$$

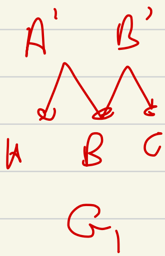
$$\begin{tikzcd}
 & A' \arrow[dr] & \\
A \arrow[ur] & & B \arrow[dl] \\
 & A'' \arrow[ul] &
\end{tikzcd} \longmapsto \;??$$

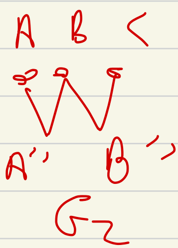




$$\left.\begin{array}{l} [A,A'] + [A',B] \\ + [B,A''] + [A'',A] \end{array}\right\} \beta$$

$= \nu_1(\sigma_1, \sigma_2)$





		∂_1
σ_1 :	$[A, A'] + [A', B]$	$[B] - [A]$
$-\sigma_2$:	$[B, A''] + [A'', A]$	$-([A] - [B])$

$$(\partial_1 \sigma_1, \partial_1 \sigma_2)$$

$$= \big([B] - [A], \; -([A] - [B]) \big)$$

$$= \big([B] - [A], \; [B] - [A] \big)$$

are equal, and $\in C_0(\sigma_1 \cap \sigma_2)$

So we can

$$\tau = [B] - [A] \in C_0(\sigma_1 \cap \sigma_2) \quad \overset{\circ}{A} \quad \overset{\circ}{B} \quad \overset{\cdot}{C}$$

So $\tau \in \mathcal{C}_c(G_1 \cap G_2)$

and

$$H_0(G_1 \cap G_2) = \mathcal{C}_0(G_1 \cap G_2) \Big/ \begin{matrix}\text{Image}(\partial_1) \\ \text{in } (G_1 \cap G_2)\end{matrix}$$

① τ exists, is it unique? Yes

② Does $\tau \in \mathcal{C}_0(G_1 \cap G_2)$ or really $\tau_\sim \in H_0(G_1 \cap G_2)$ depend on (σ_1, σ_2)?

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① What if $\tau \longmapsto (\partial_1 \sigma_1, \partial_2 \sigma_2)$

$\tau' \mapsto$

$$\mathcal{C}_0(G_1 \cap G_2) \xrightarrow{\mu_0} \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2)$$

is injective. So

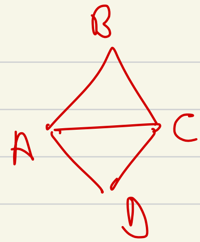
$$(\tau - \tau') \longmapsto 0 \implies \tau = \tau'$$

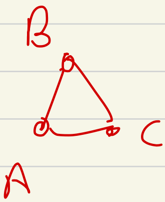
(2) Does $\tau \in \mathcal{C}_0(G_1 \cap G_2)$ or really $\tau_\sim \in H_0(G_1 \cap G_2)$ depend on (σ_1, σ_2)?

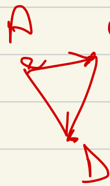
$$(\sigma_1, \sigma_2) \mapsto \beta$$
$$(\sigma_1', \sigma_2') \mapsto \beta$$

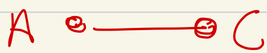
$$\begin{array}{ccccccccc}
0 \to & \mathcal{C}_1(G_1 \cap G_2) & \to & \mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2) & \to & \mathcal{C}_1(G) & \to 0 \\
& \downarrow \partial_1 & & \downarrow \partial_1 \oplus \partial_1 & & \downarrow \partial_1 & \\
0 \to & \mathcal{C}_0(G_1 \cap G_2) & \to & \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2) & \to & \mathcal{C}_0(G) & \to 0
\end{array}$$

Example 2:

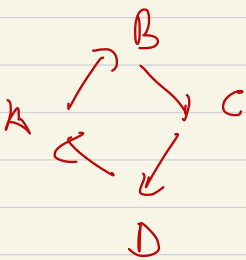
G  $\left(K^{c}_{abc}\right)$

G_1 

 G_2



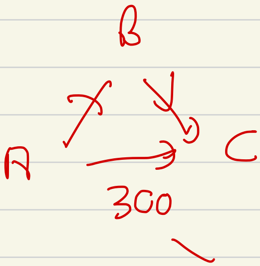
$G_1 \cap G_2$

$B \in H_1(G)$ 

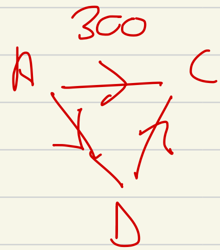
$$[A,B]+[B,C]+[C,D]+[D,A]$$

$$(\sigma_1,\sigma_2) = \big([A,B]+[B,C],\ -([C,D]+[D,A])\big)$$

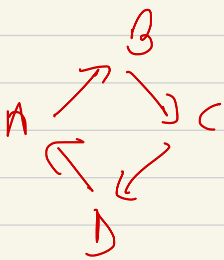
$$(\sigma_1',\sigma_2') = \Big(\underbrace{[A,B]+[B,C]+300[A,C]}_{\sigma_1'},\ \underbrace{[D,C]+[A,D]}_{\substack{300[A,C] \\ \sigma_2'}}\Big)$$



$300(A, C)$



$\sigma_1' - \sigma_2'$



$$\partial_1(\sigma_1, \sigma_2) = \mu_D(\overbrace{[C]-[A]}^{\tau}) = \left([C]-[A],\ [C]-\cancel{[A]}[A]\right)$$

$$\boxed{\begin{array}{l} \tau \neq \tau' \in \mathcal{C}_0 \\ \tau \overset{?}{=} \tau' \in H_0 \ ?? \end{array}}$$

$$\partial_1(\sigma_1', \sigma_2') = \mu_D\left(\overbrace{\begin{array}{c}[C]-[A] \\ +\,300([C]-[A])\end{array}}^{\tau'}\right) = \left(\begin{array}{c}[C]-[A]\,+ \\ 300([C]-[A])\end{array},\ \begin{array}{c}[C]-[A] \\ + \\ 300([C]-[A])\end{array}\right)$$

$$\begin{tikzcd}
\mathcal{C}_1(G_1 \cap G_2) = \mathcal{C}_1\left(\overset{A}{\circ}\!\!-\!\!-\!\!\overset{C}{\circ}\right) = \mathbb{R}([A,C]) \arrow[d, "\partial_1"'] & \left\{ \begin{array}{c} [A,C] \\ \rotatebox[origin=c]{-90}{$\mapsto$} \\ {[C]-[A]} \end{array} \right. \\
\mathcal{C}_0(G_1 \cap G_2) = \mathcal{C}_0\left(\overset{A}{\circ}\!\!-\!\!-\!\!\overset{C}{\circ}\right) = \mathbb{R}[A] + \mathbb{R}[C] &
\end{tikzcd}$$

Alternate (general case) proof!

$$(\sigma_1, \sigma_2) \longmapsto \beta$$
$$(\sigma_1', \sigma_2') \longmapsto \beta$$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}_1(G_1 \cap G_2) \arrow[r] \arrow[d, "\partial_1"] & \mathcal{C}_1(G_1) \oplus \mathcal{C}_1(G_2) \arrow[r] \arrow[d, "\partial_1 \oplus \partial_1"] & \mathcal{C}_1(G) \arrow[r] \arrow[d, "\partial_1"] & 0 \\
0 \arrow[r] & \mathcal{C}_0(G_1 \cap G_2) \arrow[r] & \mathcal{C}_0(G_1) \oplus \mathcal{C}_0(G_2) \arrow[r] & \mathcal{C}_0(G) \arrow[r] & 0
\end{tikzcd}$$

$$\tau \longmapsto (\partial \sigma_1, \partial \sigma_2)$$
$$\tau' \longmapsto (\partial \sigma_1', \partial \sigma_2')$$

(where 3UO[A,C] was created)

$$\alpha \longmapsto (\sigma_1,\sigma_2)-(\sigma_1',\sigma_2') \longmapsto 0$$

$$\begin{tikzcd}
0 \arrow[r] & C_1(G_1\cap G_2) \arrow[r] \arrow[d, "\partial_1"] & C_1(G_1)\oplus C_1(G_2) \arrow[r] \arrow[d, "\partial_1\oplus\partial_1"] & C_1(G) \arrow[r] \arrow[d, "\partial_1"] & 0 \\
0 \arrow[r] & C_0(G_1\cap G_2) \arrow[r] & C_0(G_1)\oplus C_0(G_2) \arrow[r] & C_0(G) \arrow[r] & 0
\end{tikzcd}$$

$$(\partial_1\alpha) \longmapsto (\partial_1\sigma_1,\partial_1\sigma_2)-(\partial_1\sigma_1',\partial_1\sigma_2')$$

$$\tau-\tau' \longmapsto$$

So

$$\tau-\tau' = \partial\alpha$$

$$\tau \equiv \tau' \mod \text{Image } \partial_1(G_1\cap G_2)$$

$$\Rightarrow \tau_\sim = \tau'_\sim \in H_0(G_1\cap G_2)$$