CPSC 531F February 14, 2025

(Reading break: Feb 17-21)

① Finish Mayer-Vietoris:

$$\begin{tikzcd}
0 \arrow[r] & H_1(G_1\cap G_2) \arrow[r] & H_1(G_1)\oplus H_1(G_2) \arrow[r] & H_1(G) \arrow[dll, rounded corners, "\delta"'] & \\
 & H_0(G_1\cap G_2) \arrow[r] & H_0(G_1)\oplus H_0(G_2) \arrow[r] & H_0(G) \arrow[r] & 0
\end{tikzcd}$$

Thm: This sequence is exact.

② Mayer-Vietoris for complexes:

(2a) State theorem

(2b) Apply to suspensions

Mayer-Victoris for complexes:

Let $K_1, K_2 \subset K$ abstract simplicial complexes with $K_1 \cup K_2 = K$, and $\dim(K) = d$. Then there is exact sequence

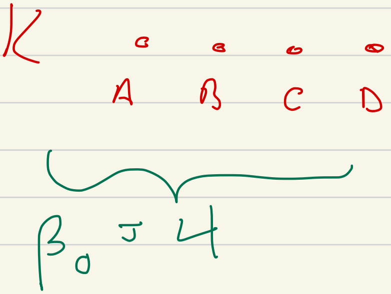
$$\begin{tikzcd}
0 \arrow[r] & H_d(K_1 \cap K_2) \arrow[r] & H_d(K_1) \oplus H_d(K_2) \arrow[r] & H_d(K) \arrow[dll, "\delta_d"'] & \\
 & H_{d-1}(K_1 \cap K_2) \arrow[r] & H_{d-1}(K_1) \oplus H_{d-1}(K_2) \arrow[r] & H_{d-1}(K) \arrow[dll, "\delta_{d-1}"'] & \\
 & \cdots & & \cdots \arrow[dll, "\delta_1"'] & \\
 & H_0(K_1 \cap K_2) \arrow[r] & H_0(K_1) \oplus H_0(K_2) \arrow[r] & H_0(K) \arrow[r] & 0
\end{tikzcd}$$

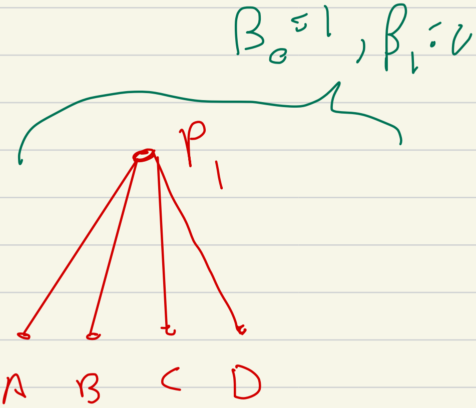
Let K be abst simp complex,

and $P_0, P_1 \notin$ Vertices of K. Then

$$\text{Suspension}_{P_1, P_2}(K) \overset{def}{=} \text{Cone}_{P_1}(K) \cup \text{Cone}_{P_2}(K)$$

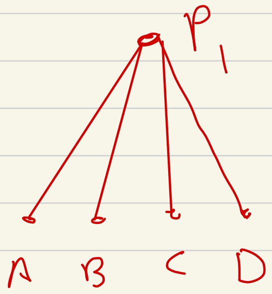
Examples:





$\text{Cone}_{P_1}(K)$

Examples:

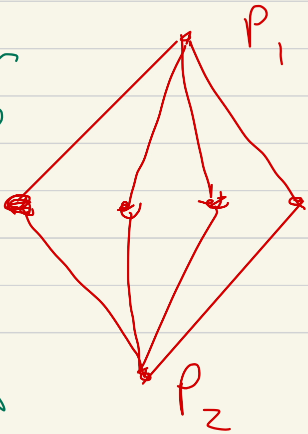


$$\beta_i\left(\text{Cone}_{p_1}(K) \right) = \begin{cases} 1 & i=0 \\ 0 & i\geq 1 \end{cases}$$

$$\beta_i(L) \overset{\text{def}}{=} \dim(H_i(L))$$

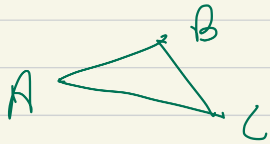
$\beta_0 = 1$

$\beta_1 = 3$



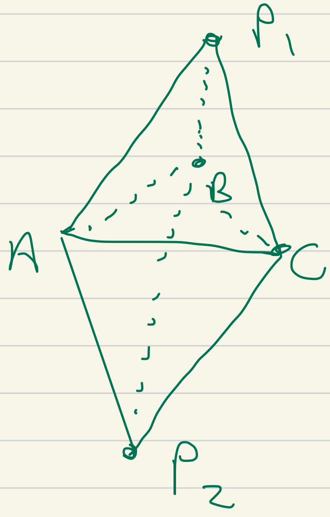
$\text{Suspension}_{p_1,p_2}(K)$

$$K = \left\{ \begin{array}{l} \text{Cone}_{p_1}(K) \\ \cup\ \text{Cone}_{p_2}(K \end{array} \right.$$



$\beta_0 = 1$ ← K

$\beta_1 = 1$



$\}$ $\mathrm{Cone}_{p_1}(K)$

$\}$ $\mathrm{Cone}_{p}(K)$

Claim: $L = \mathrm{Susp}_{p_1, p_2}(K)$

$\beta_0(L) = 1$, $\beta_1(L) = 0$, $\beta_2(L) = 1$

$$\partial_2\Big([P_1, A, C] + [P_1, C, B] + [P_1, B, A]\Big)$$

$$= (A, C) + [C, B] + [B, A]$$

$$\partial_2\Big(\text{ put } P_2 \text{ for } P_1 \Big)$$

= same boundary

subtract! get difference

$\tau \in \mathcal{C}_2(K)$, τ written as sum σ, 2-faces

$$\partial_2 \tau = 0$$

After break: "topologically"

Susp(2-Sphere) = 3-Sphere

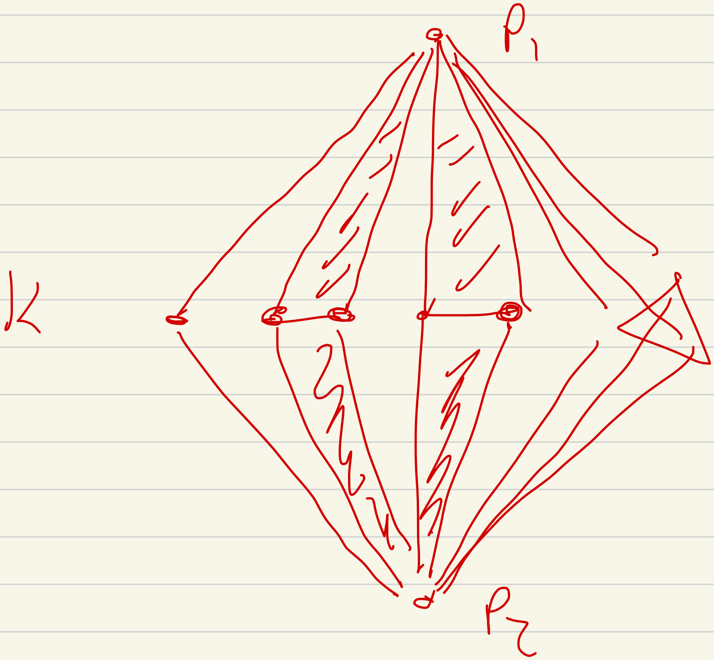
" 3-Sphere = 4-Sphere

Thm:

$$\beta_0(\mathrm{Susp}(K)) = 1$$

$$\beta_1(\mathrm{Susp}(K)) = \beta_0(K) - 1$$

$i \geq 2$, $\beta_i(\mathrm{Susp}(K)) = \beta_{i-1}(K)$



(1) $\text{Cone}_p(K)$ has $H_0(\ ) \cong \mathbb{R}$

$i \geq 1 \quad H_i(\ ) = 0$

$L = \text{Susp}_{p_1, p_2}(K)$

$L_1 = \text{Cone}_{p_1}(K), \; L_2 = \text{Cone}_{p_2}(K)$

$$H_i(L_j) = \begin{cases} \mathbb{R} & i = 0 \\ 0 & i \geq 1 \end{cases} \qquad j = 1, 2$$

$L_1 \cup L_2 = L =$ suspension (K)

$L_1 \cap L_2 = K$

$$\begin{tikzcd}
0 \arrow[r] & H_d(K_1\cap K_2) \arrow[r] & H_d(K_1)\oplus H_d(K_2) \arrow[r] & H_d(K) \arrow[dll, "\delta_d"'] & \\
& H_{d-1}(K_1\cap K_2) \arrow[r] & H_{d-1}(K_1)\oplus H_{d-1}(K_2) \arrow[r] & H_{d-1}(K) \arrow[dll, "\delta_{d-1}"'] & \\
& \cdots & & \cdots \arrow[dll, "\delta_1"'] & \\
& H_0(K_1\cap K_2) \arrow[r] & H_0(K_1)\oplus H_0(K_2) \arrow[r] & H_0(K) \arrow[r] & 0
\end{tikzcd}$$

is exact $K \rightsquigarrow L$

$$\begin{tikzcd}
0 \arrow[r] & H_d(K) \arrow[r] & H_d(L_1) \oplus H_d(L_2) \arrow[r] & H_d(L) \arrow[dll] & \\
& H_{d-1}(K) \arrow[r] & H_{d-1}(L_1) \oplus H_{d-1}(L_2) \arrow[r] & H_{d-1}(L) \arrow[dll] & \\
& {} & & \cdots \arrow[dll] & \\
& H_0(K) \arrow[r] & H_0(L_1) \oplus H_0(L_2) \arrow[r] & H_0(L) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots & 0 \arrow[r] & H_3(L) \arrow[dll] & \\
H_2(K) \arrow[r] & 0 \arrow[r] & H_2(L) \arrow[dll] & \\
H_1(K) \arrow[r] & 0 \arrow[r] & H_1(L) \arrow[dll] & \\
H_0(K) \arrow[r, hook] & \mathbb{R} \oplus \mathbb{R} \arrow[r] & \mathbb{R} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & H_2(L) \arrow[dll, bend left] \\ H_1(K) \arrow[r] & 0 \end{tikzcd}$$

in an exact sequence:

$$0 \xrightarrow{\text{inject}} A \xrightarrow{\varphi_1} B \xrightarrow{\varphi_2} 0$$

is exact

Im = 0

surjective

ker

$0 \xrightarrow{\text{Image}} A = 0$

$= \ker A \to B$

$B \to 0$

is B

exactness

$\ker \varphi_2 = B$

$A \cong \text{Image } \varphi_1$

So

$$\begin{tikzcd} & 0 \arrow[r] & H_{i+1}(L) \arrow[dll, out=-30, in=150] \\ H_i(K) \arrow[r] & 0 & \end{tikzcd}$$

for all $i \geq 1$

$$\Longrightarrow H_{i+1}(L) \cong H_i(K)$$

$i+1 \geq 2$ tells us $H_{i+1}(L)$

$\uparrow$

suspective

$H_0(L) \cong \mathbb{R}$

$$\begin{tikzcd} & & 0 \arrow[r] & H_1(L) \arrow[dlll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=2ex]\tikztotarget.north) \tikztonodes -- (\tikztotarget)}] \\ H_0(K) \arrow[r] & \mathbb{R}^2 \arrow[r, "\text{image}"] & \mathbb{R} \arrow[r, "\text{ker}"] & 0 \end{tikzcd}$$

image

all of $\mathbb{R}$

ker

all of $\mathbb{R}$

ker

surjective

$$\mathbb{R}^2 \longrightarrow \mathbb{R}$$

$\mathcal{L}: A \to B$ map of vector spaces:

$$\text{Rank}(\mathcal{L}) \overset{\text{def}}{=} \dim(\text{Image}(\mathcal{L}))$$

$$= \dim(f(A))$$

So

Thm: $\dim(A) = \dim(\ker(\mathcal{L})) + \operatorname{rank}(\mathcal{L})$

$$\mathcal{L}: \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\operatorname{Image}(\mathcal{L}) = \mathbb{R}$$

$$\operatorname{rank}(\mathcal{L}) = 1$$

$$\dim(\ker(\mathcal{L})) = 1$$

$$\begin{tikzcd} H_d(K) \arrow[r] & \mathbb{R}^2 \arrow[r, "\mathcal{L}"] & \mathbb{R} \end{tikzcd}$$

image has to be 1-dim

$\ker \mathcal{L}$ is 1-dim

$$\begin{tikzcd}
 & 0 \arrow[r] & H_1(L) \arrow[dll, bend left] \\
H_0(K) \arrow[r] & {} \arrow[d, mapsto] & \\
 & \text{ker dim: } \dim(H_0(K)) - 1 &
\end{tikzcd}$$

$$\dim\left(H_1(L)\right) = \dim\left(H_0(K)\right) - 1$$