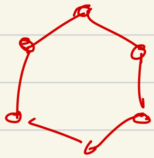
CPSC 531F Feb 24, 2025

Last time: used Mayer-Vietoris to compute simplicial homology groups $H_i^{simp}(K)$ for various abstract simplicial complexes, K, and

$$\beta_i^{simp}(K) = \dim\left(H_i^{simp}(K)\right)$$

Today: Start to think "geometrically" to better understand $\beta_i^{simp}(K)$

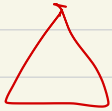
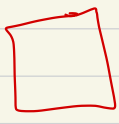
Example 1:

$\beta_i^{simp}\Big($  $\Big) = \begin{cases} 1 & i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$

similarly for any cycle.

Why the same for any ż cycles?

Ans:

 ,  ,  , ... $\subset \mathbb{R}^2$

are all homeomophic as subsets of $\mathbb{R}^2$

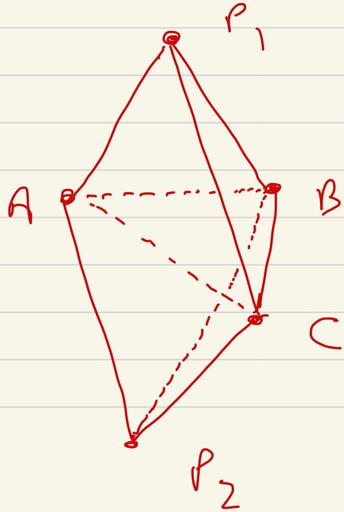
Example 2: The following all have

$\beta_0 = \beta_2 = 1$, $\beta_i = 0$ if $i \neq 0, 2$:

(A)

"3-simplex minus its interior"

(B) the suspension of any cycle!



All homeomorphic

The plan:

- Define when $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are homeomorphic.

- Develop intuition for topological spaces.

- Eventually: define singular homology and

$$H_i^{simp}(K) \simeq H_i^{sing}(|K|)$$

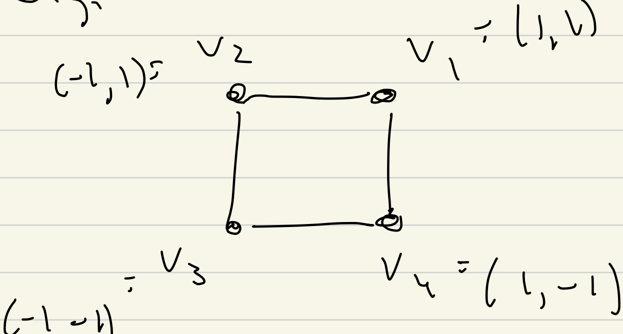
Recall:

<u>simplex</u> a set

$$\mathrm{Conv}(\vec{a}_0, \dots, \vec{a}_d) \subset \mathbb{R}^n$$

where $\vec{a}_0, \dots, \vec{a}_d \in \mathbb{R}^n$ are in general position.

E.g.



$\subset \mathbb{R}^2$

$$K = \{ \emptyset, \{v_1\}, \dots, \{v_4\}, \mathrm{Conv}(v_1, v_2), \mathrm{Conv}(v_2, v_3), \dots, \mathrm{Conv}(v_4, v_1) \}$$

A *simplicial complex* K in $\mathbb{R}^n$ is a set of simplices in $\mathbb{R}^n$ s.t.

(1) if $X \in K$, then any face of X also lies in K

$\Big($ $X = \mathrm{Conv}(a_0, \dots, a_d) \in K \Rightarrow$

$A' \subset \{a_0, \dots, a_d\}$,

$\mathrm{Conv}(A')$ is a *face* of X

and $\mathrm{Conv}(A') \in K$ $\Big)$

(2) If $X, X' \in K$ then $X \cap X'$ is a face of both X, X'

$$K^{abs} = \{ \emptyset, \{v_1\}, \dots, \{v_4\}, \{v_1, v_2\}, \dots$$
$$\dots \{v_4, v_1\} \}$$

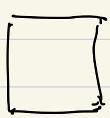
combinatorial structure

Also defined:

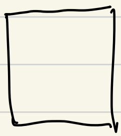
$$|K| = |K|_{geom}$$

$$= \bigcup_{X \in K} X \subset \mathbb{R}^2$$

In the exampl

$$|K| =$$  $$\subset \mathbb{R}^2$$

Claim!

 $\subset \mathbb{R}^2$

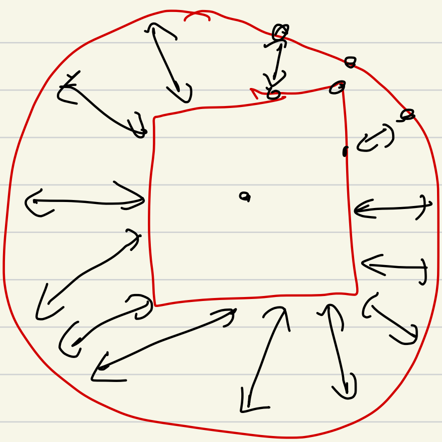
and

$$S^1 = \{(x,y) \mid x^2+y^2=1\} \subset \mathbb{R}^2$$

 $\subset \mathbb{R}^2$

are "topologically the same"

Why are □ and ○ <u>the same</u>, or <u>homeomorphic</u>



There's a (one-to-one) bijection

$f: \bigcirc \longrightarrow \square$

s.t. f and f^{-1} are continuous

Some $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$

and

$$f : X \longrightarrow Y$$

is <u>continuous</u> at $x_0 \in X$ if

$$\lim_{x \to x_0} f(x) = f(x_0) \quad (*)$$

What does $(*)$ "limit" mean

" We can get $f(x)$ ~~as~~ close to $f(x_0)$ as we like, by taking x sufficiently close to x_0 "

i.e.,

For any $\varepsilon > 0$, there is a $\delta > 0$

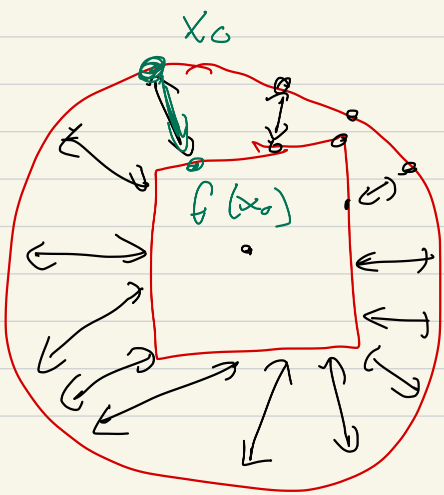
s.t.

$$|f(x_0) - f(x)| \le \varepsilon$$

provided that

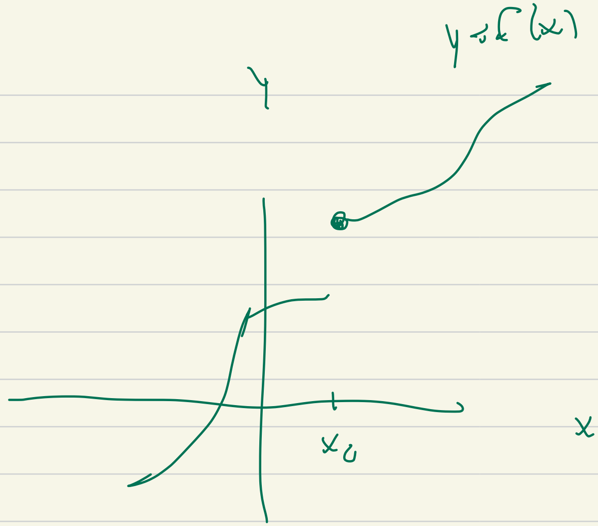
$$|x_0 - x| \le \delta \qquad (\text{and } x \in X)$$

$(*)$

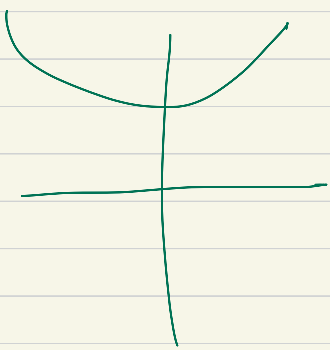


Rem:

$f: \mathbb{R} \to \mathbb{R}$



not continuous



is continuous

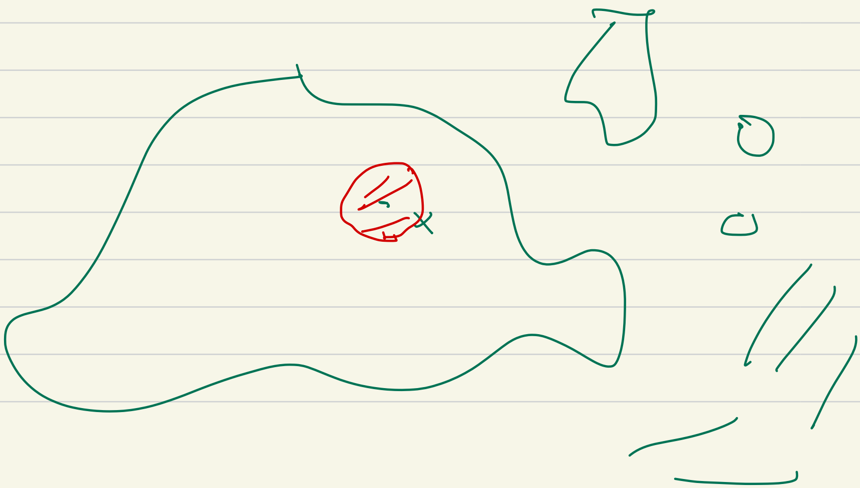
If $x \in \mathbb{R}^n$, then a neighbourhood of x is a subset $N \subset \mathbb{R}^n$ s.t. for some $\rho > 0$, we have

$$\mathrm{Ball}^{\text{closed}}_{\rho}(x) \subset N$$

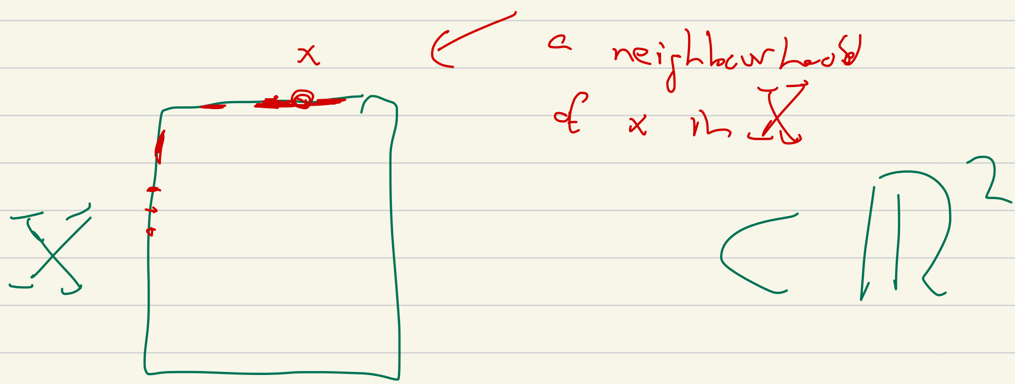
where

$$\mathrm{Ball}^{\text{closed}}_{\rho}(x) \overset{\text{def}}{=} \{x' \mid |x' - x| \le \rho\}$$

N !



Also, if $X \subset \mathbb{R}^n$, and $x \in X$, then a <u>neighbourhood of x</u> <u>in X</u> (with respect viewing X as contained in $\mathbb{R}^n$) is any set of the form $X \cap N$ where N is a neighbourhood of $x \in \mathbb{R}^n$,



Nbhd(x)

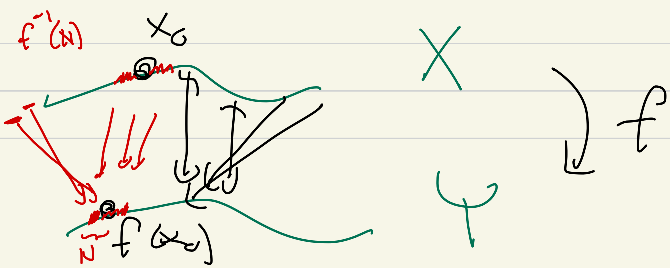
There is a textbook by Armstrong that takes this pedagogical approach (Chapters 1 & 2):

(**) is equivalent to:

if N is any neighbourhood of $f(x_0)$, then

$$f^{-1}(N) = \{ x \mid f(x) \in N \}$$

is a neighborhood of x_0



So

$$\begin{array}{ccc} f: X & \longrightarrow & Y \\ \cap & & \cap \\ \mathbb{R}^n & & \mathbb{R}^m \end{array}$$

is continuous if for every x_0, every neighborhood, N, of $f(x_0)$ has

$$f^{-1}(N)$$

a neighborhood of x_0.

Claim: $X \xrightarrow{f} Y \xrightarrow{g} Z$

f, g continuous, $g \circ f$ is continuous

Next time: <u>even</u> <u>simpler</u> !

we'll look at

$\left\{ \begin{array}{l} \text{open subsets} \\ \\ \text{open neighbourhoods} \end{array} \right\}$