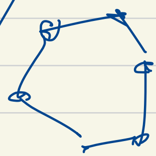
CPSC 531F Feb 26, 2024

Algebraic topology!

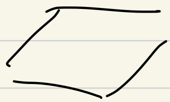
(1) Claim:

Betti$_1$ () = independent of the cycle length

Principle: Any 2 simplicial complexes that are homeomorphic have the same homology groups & Betti numbers.

(2) Any cone:

K



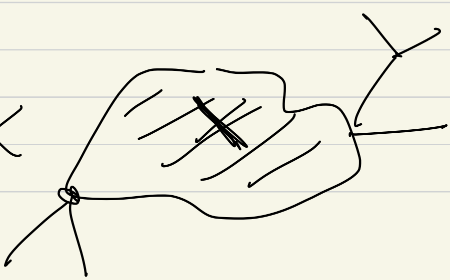
P

$$\mathrm{Betti}_i\left(\begin{matrix}\text{Any}\\ \text{cone}\end{matrix}\right) = \mathrm{Betti}_i(\cdot)$$

$$= \begin{cases} 1 & \text{if } i=0 \\ 0 & \text{otherwise} \end{cases}$$

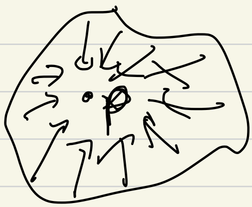
<u>Definition</u> $X \subset \mathbb{R}^n$

X



We say that X is <u>contractible</u>

if there's a continuous map:

X 

$$X \times \underbrace{[0,1]}_{\text{real interval}} \to X$$

s.t. $$X \times \{0\} \xrightarrow{\text{identity}} X$$

$$X \times \{1\} \longrightarrow \bullet\, p$$

More formally, there is $p \in X$ and a map (continuous)

$$\begin{array}{ccccc} f: & X & \times\ [0,1] & \longrightarrow & X \\ & \cap & \cap & & \cap \\ & \mathbb{R}^n & \times\ \mathbb{R} & & \mathbb{R}^n \end{array}$$

$$\underbrace{\mathbb{R}^n \times \mathbb{R}}_{\substack{\cap \\ \mathbb{R}^{n+1}}}$$

s.t.

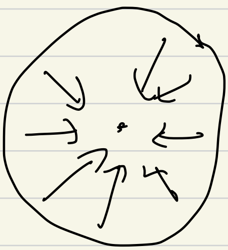
$$f(x,0) = x$$
$$f(x,1) = p$$

Example 1: Ball of radius 1

about $(0,0)$ in $\mathbb{R}^2$

$$\{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 1\} = X,$$

$p = (0,0)$

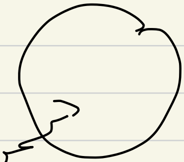


$$f(\vec{x}, t) = \vec{x}(1-t)$$

Example 2:

$$S^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$$

Example 2

$S^1 =$  $\subset \mathbb{R}^2$

without interior

$(\beta_1 = 1$ not $0)$

p

$S^1 = X$, $p \in X$

S^2 :  $\subset \mathbb{R}^3$

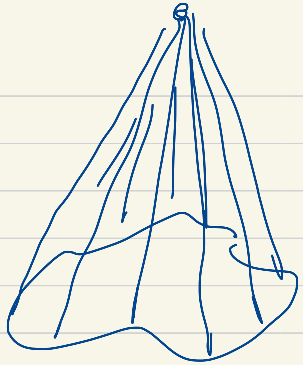
p

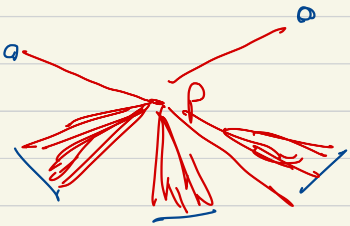
Example 3

$(\beta_2 = 1,$ not $0)$

Example 4:

X = A Cone :





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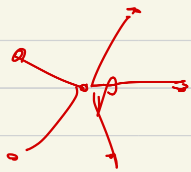


, etc

Claim: Any "cone" is contractible...

Back to "thinking geometrically" and "topological spaces"

For $\vec{x} \in \mathbb{R}^n$, a neighborhood of $\vec{x}$ is a set $N \subset \mathbb{R}^n$ s.t.

$$\mathrm{Ball}_\rho(x) \subset N$$

for some $\rho > 0$,

$$\mathrm{Ball}_\rho^{\text{closed}}(x) = \{ x' \in \mathbb{R}^n \mid \| x - x' \| \leq \rho \}$$

$$\mathrm{Ball}_\rho^{\text{open}}(x) = \{ x' \in \mathbb{R}^n \mid | x - x' | < \rho \}$$

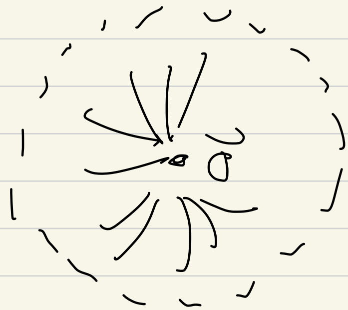
A subset $U \subset \mathbb{R}^n$ is open

if

(1) for every $\vec{x} \in U$, U is a

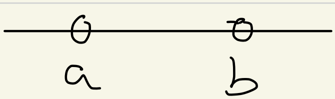
neighborhood of $\vec{x}$

e.g.

$B_3^{open}(\vec{0}) =$ 

is open,

$(a, b) \subset \mathbb{R}$ open interval, $a < b$



open

and

$$B_3^{closed}(\vec{0}) =$$

and

$[a, b]$ closed interval

not open

__Theorem__:

If $f : \mathbb{R}^n \to \mathbb{R}^m$, then

- f is continuous iff
- for every open subset $U \subset \mathbb{R}^m$,

  $f^{-1}(U) \subset \mathbb{R}^n$ is open

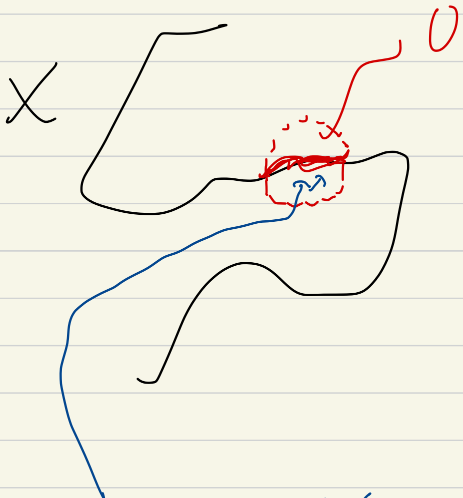
Thm:
$$\begin{array}{ccccc} f : & X & \longrightarrow & Y & \\ & \cap & & \cap & \\ & \mathbb{R}^n & & \mathbb{R}^m & , \end{array}$$

we say $W \subset X$ is __relatively open in X__ (as $X \subset \mathbb{R}^n$)

means

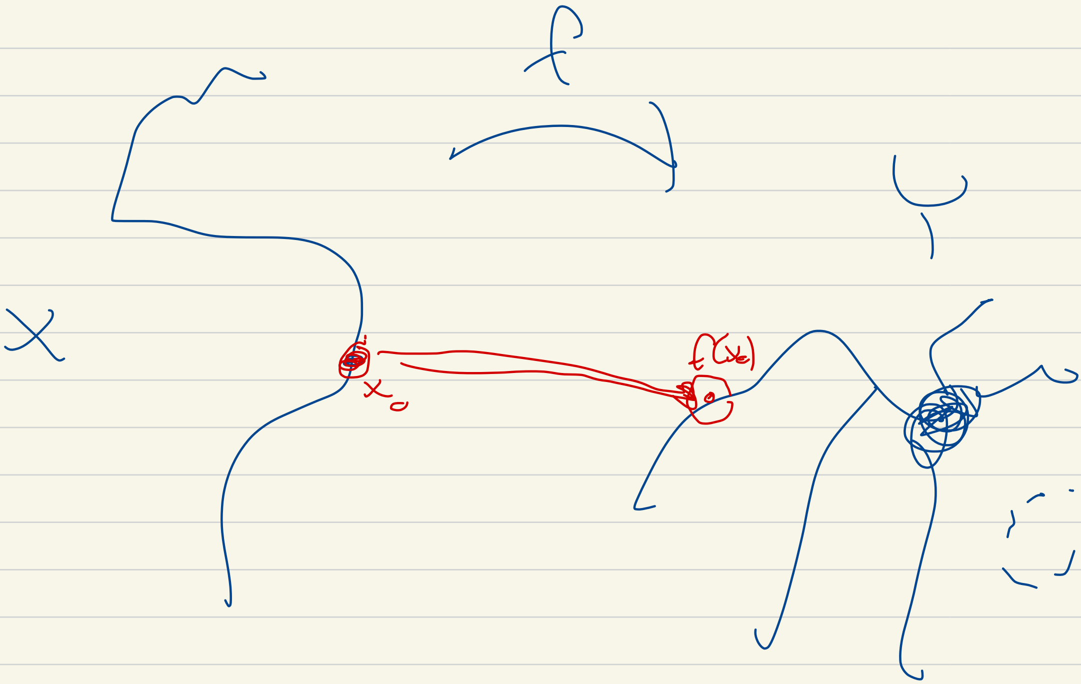
W can be written as $X \cap U$

where U is open in $\mathbb{R}^n$



$\subset \mathbb{R}^2$

$W = X \cap U$



f is continuous iff

for any relatively open set

W_Y, $f^{-1}(W_Y)$ is

open in X.

<u>Def</u>: A topological space is a pair $(X, \mathcal{O})$ where

- X is a set
- $\mathcal{O}$ is a collection of subsets of X (we call "open sets")

such that:

(1) $\emptyset, X \in \mathcal{O}$

$\emptyset, X$ are open

(2) if $U_1, U_2 \in \mathcal{O}$, $U_1 \cap U_2 \in \mathcal{O}$

(3) If $\{U_i\}_{i \in I}$ and each $U_i \in \mathcal{O}$, then

$$\bigcup_{i \in I} U_i \in \mathcal{O}$$

=

Claim:

- Open subsets of $\mathbb{R}^n$ have properties (1) - (3)

Exercise: $X \subset \mathbb{R}^n$, then

$\{$ relatively open subset of $X \}$

also (1) – (3)