CPSC 531F Feb 28

Examples of topological spaces

So far:

(1) $\mathbb{R}^n$, open sets

(2) $X \subset \mathbb{R}^n$, relatively open sets

e.g.

$$S^n = \{ \vec{x} \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1 \}$$

Topological Space: $(X, \mathcal{O})$ s.t.

(1) X is a set

(2) $\mathcal{O}$ a collection of subsets of X

closed under
- finite intersections
- arbitrary unions.

$Z \subset X$ is <u>closed</u> if $X \setminus Z$ is open

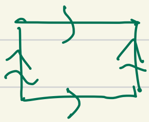
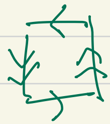
$W \subset X$:

- closure of W
- boundary of W
- interior of W

(1) $\mathbb{RP}^n$ = real n-dim projective space

(2) T^d = d-dim torus

(3) $S^{n_1} \vee S^{n_2} \vee \_\_ \vee S^{n_k}$ wedge sum of spheres

(4) T^2 =  $\mathbb{RP}^2$ = 

New spaces from old ones X, Y

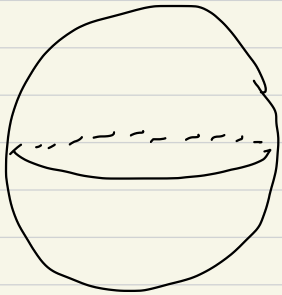
(1) $X' \subset X$ (2) $X \times Y$ (3) $X/\sim$ equiv relation

(4) $X \amalg Y$ disjoint union

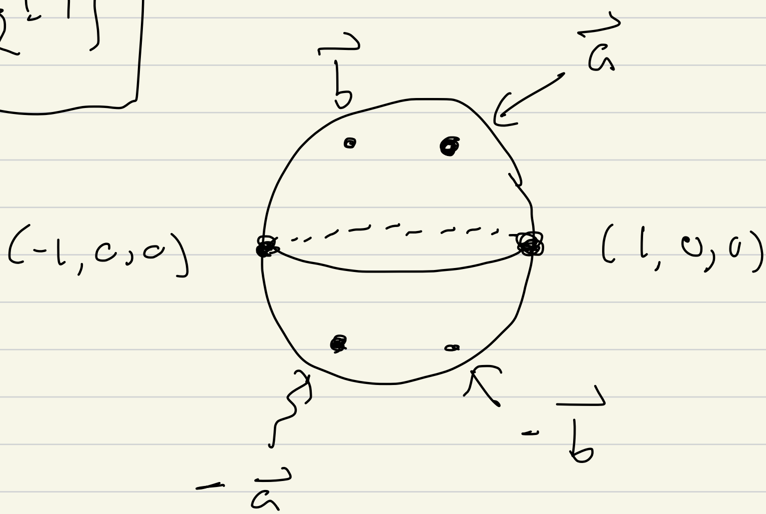
(5) $X \vee Y$ wedge sum

$\mathbb{RP}^2 =$ "$S^2/\{\pm 1\}$" $=$ lines in $\mathbb{R}^3$

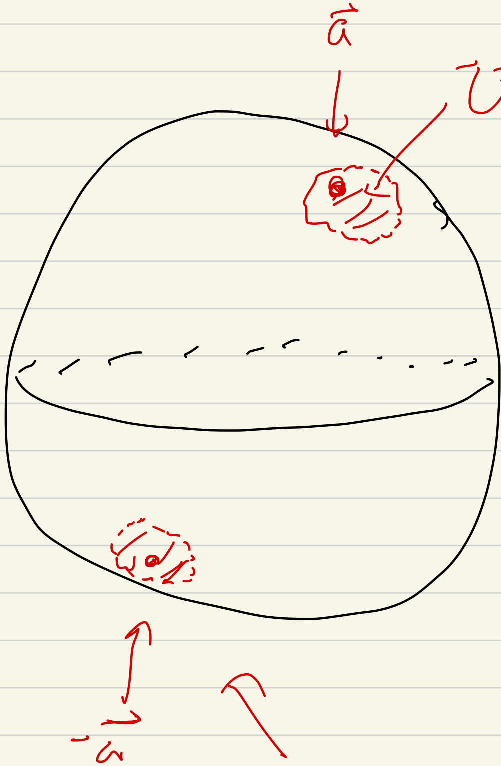
S^2



$\boxed{\text{"}S^2/\{\pm 1\}\text{"}}$



open subset of $S^2/\{\pm 1\}$



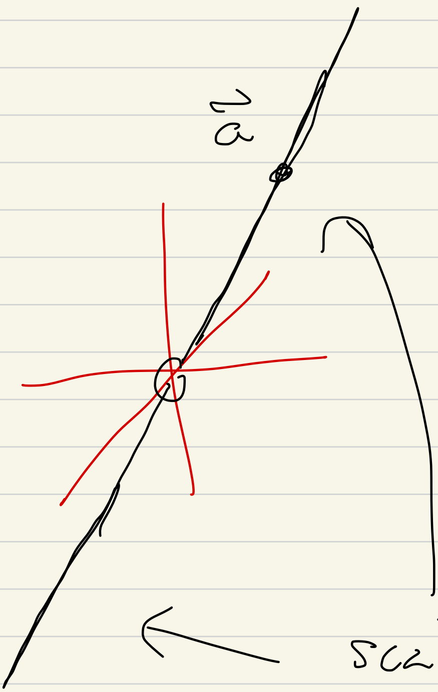
$\bar{U} = S^2 \cap W$

open in $\mathbb{R}^3$

$-U = \{-\vec{u} \mid \vec{u} \in \bar{U}\}$

$\mathbb{RP}^2$: "lines thru $(0,0,0)$ in $\mathbb{R}^3$"

$$\text{"} \; \mathbb{R}^3 \setminus \{(0,0,0)\} \Big/ (\mathbb{R}\setminus\{0\}) \; \text{"}$$



$\vec{a} \sim \alpha \vec{a}$,

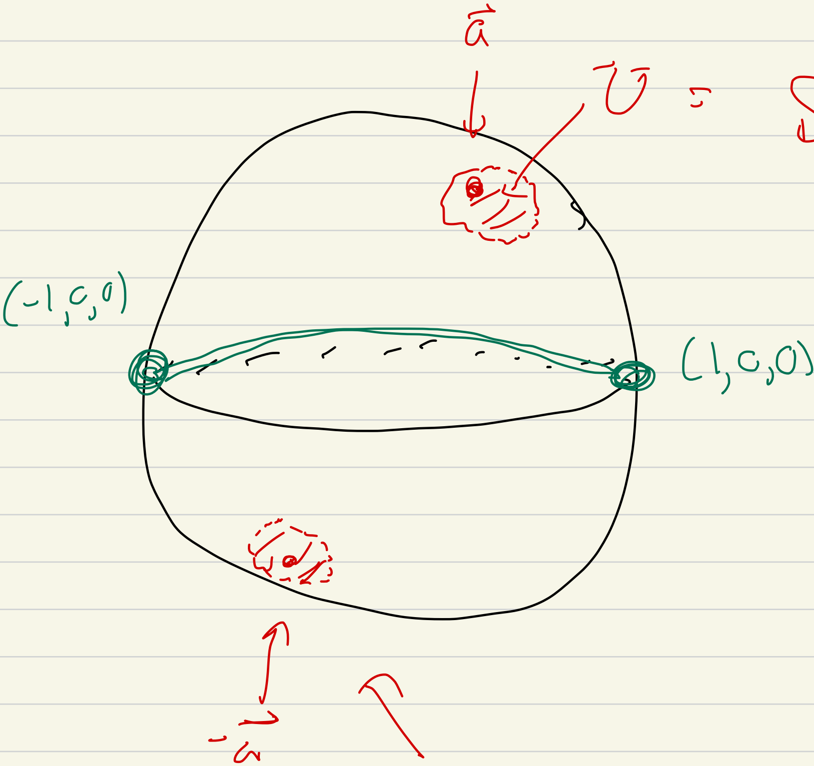
$\alpha \in \mathbb{R}\setminus\{0\}$

scalar multiples of $\vec{a}$

"S^2 homeomorphic to $\simeq$ $S^2/\{\pm 1\}$??"

$U = S^2 \cap W$

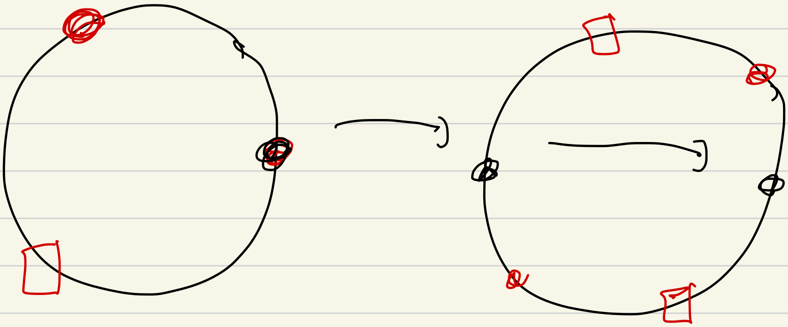
open in $\mathbb{R}^3$



$$-U = \{ -\vec{u} \mid \vec{u} \in U \}$$

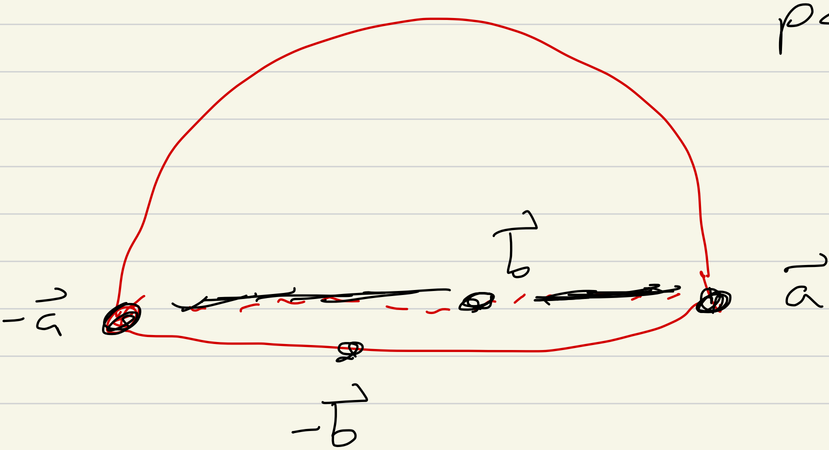
$$S^2 = \{ (x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1 \}$$

Rem! $S^1 \simeq S^1/\{\pm 1\}$



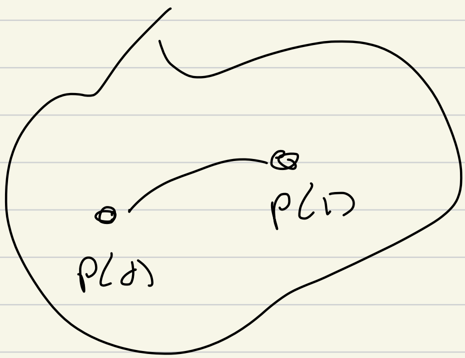
$S^2/\{\pm 1\}$

"path"



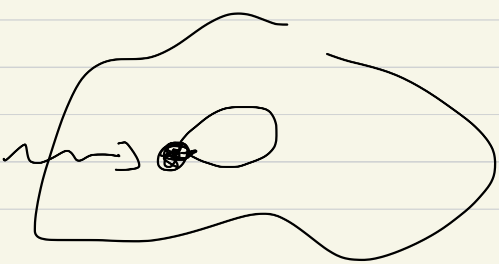
A path in a topological space:

$$p : [0,1] \longrightarrow X$$

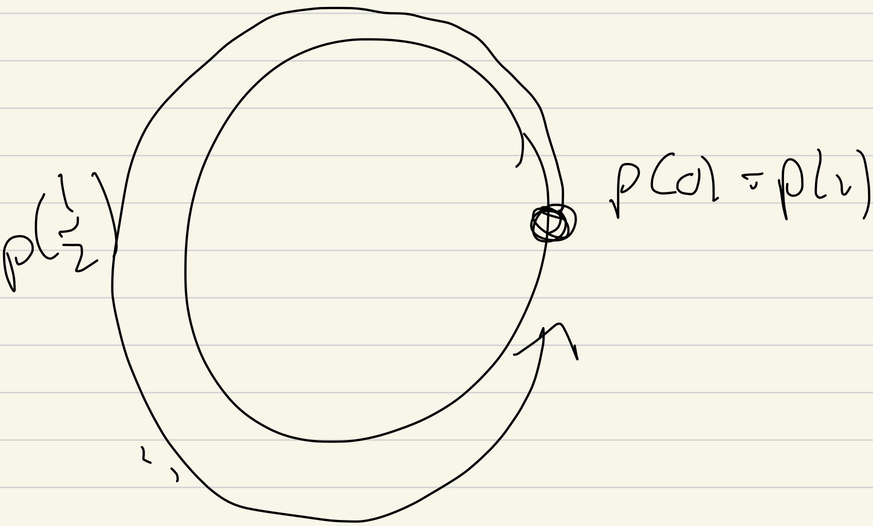
$[0,1] \longrightarrow$ 

A <u>closed</u> <u>path</u> $\quad p(1) = p(0)$

$p(1)$
$= p(0)$



Closed path on S^1 !



$$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{\vec{0}\} \Big/ \mathbb{R} \setminus \{0\}$$

$$= S^n \Big/ \{\pm 1\}$$

real projective n-dim space

Let $(X, \mathcal{O})$ be a topological space. An <u>equivalence relation</u> on X is a relation (formally $R \subset X \times X$ we write $x_1 \sim x_2$ to mean $(x_1, x_2) \in R$)

s.t.

① $x \sim x$, ② $x_1 \sim x_2 \Rightarrow x_2 \sim x_1$

③ $x_1 \sim x_2,\ x_2 \sim x_3 \Rightarrow x_1 \sim x_3$

Eg.

$$\mathbb{Z} = \{ \ldots, -1, 0, 1, 2 \}$$

$x_1 \sim x_2 \iff x_1 - x_2$ is divisible by 2

$0 \sim 2 \sim 4 \sim 6 \sim -2 \sim \ldots$

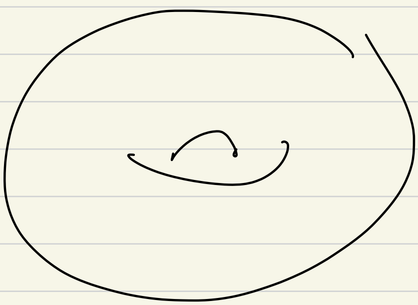
$1 \sim 3 \sim -1 \sim 5 \sim -3 \sim \ldots$

$\Rightarrow \mathbb{Z} / 2\mathbb{Z}$

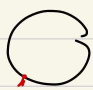
S^2 ! $s_1 \sim s_2$

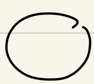
$\Leftrightarrow \quad s_1 = \pm s_2$

$T^2 = S^1 \times S^1$



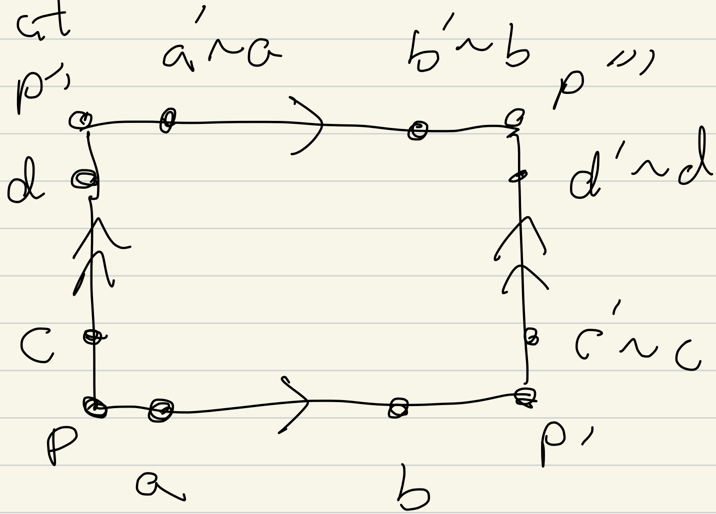
2-dim in $\mathbb{R}^3$

S^1  $\subset \mathbb{R}^2$

S^1  $\subset \mathbb{R}^2$

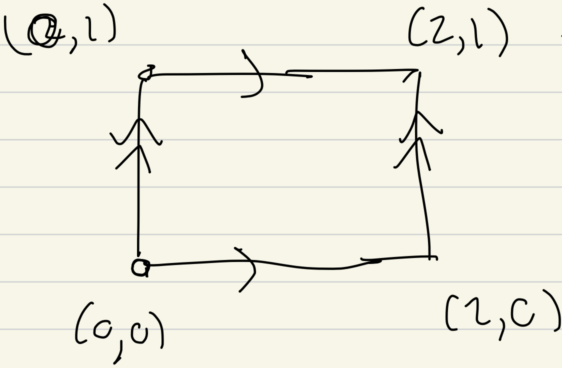
$$\left.\begin{matrix} S^1 \subset \mathbb{R}^2 \\ S^1 \subset \mathbb{R}^2 \end{matrix}\right\} \quad S^1 \times S^1 \subset \mathbb{R}^4$$

Look at



$\mathbb{R}^2$

$p \sim p' \sim p'' \sim p'''$



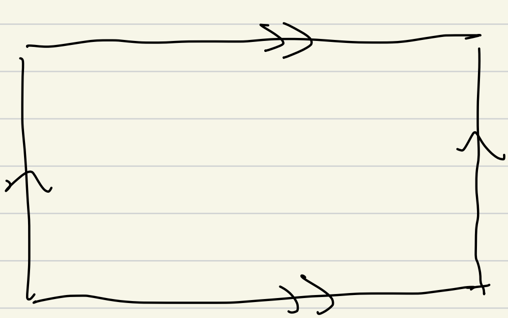
$X = [0,2] \times [0,1]$

Say $(X, \mathcal{O})$ is a topological space, $\sim$ equiv relation on X : "quotient space" :

$$\Big(\underbrace{X/\sim}_{\text{equivalence classes}} , \ \mathcal{O}_\sim \Big)$$

$U \subset X/\sim$ is open (in $\mathcal{O}_\sim$)

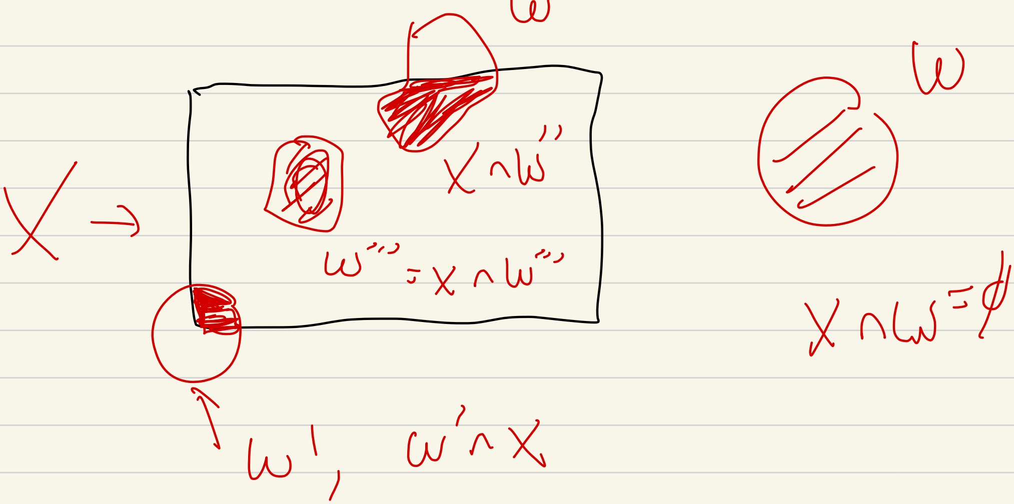
iff ! $U = W/\sim$ s.t. W is open in X



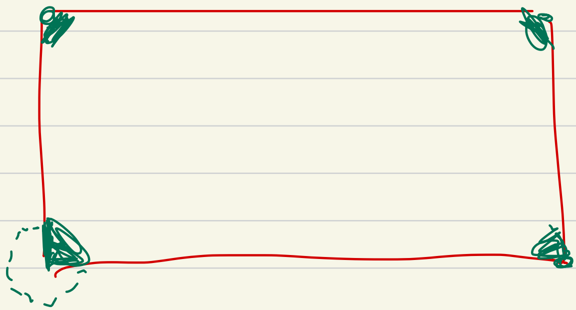
$X/\sim$

relatively open subsets

$[0,2] \times [0,1]$ in $\mathbb{R}^2$



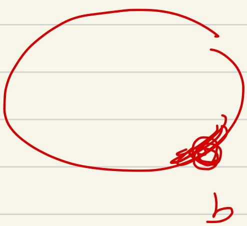
$X/\sim$



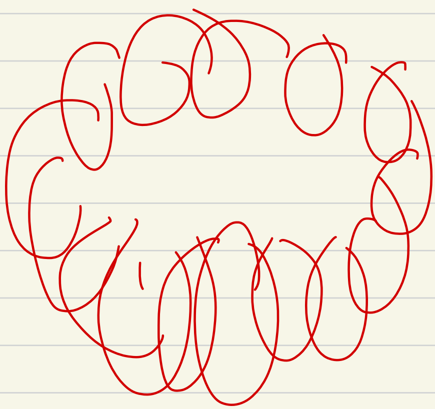
U has to consist of equiv classes

S^1

a, b



S^1



$S^1 \times S^1$