CPSC 531F March 3, 2025

Goal: Define continuous maps

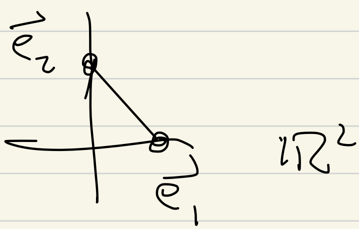
$$\Delta^n \longrightarrow X$$

any topological space

A single, fixed n-simplex,

$$\Delta^n = \mathrm{Conv}(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_{n+1}) \subset \mathbb{R}^{n+1}$$

Δ^1 :



$\vec{e}_1 = (1, 0, 0, \dots)$, $\vec{e}_2 = (0, 1, 0, \dots)$, ...

$\vec{e}_i$ = ith standard basis vector in $\mathbb{R}^{n+1}$

Use maps

$$\Delta^n \to X$$

(1) to define singular homology groups,

$$H_i^{sing}(X)$$

defined for any topological space.

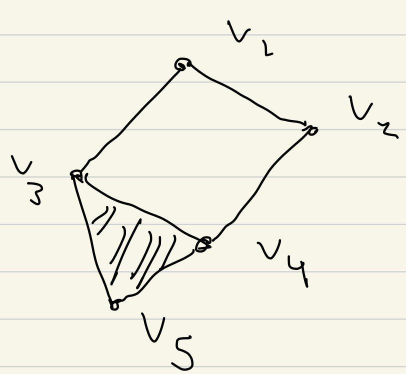
The singular homology groups have a number of advantages over simplicial homology, but the two will agree on simplicial complexes.

(2) We use maps

$$\Delta^n \to X$$

to define a "Δ-complex structure on X", which is a generalization of simplicial complexes.

Simplicial complex, in $\mathbb{R}^n$



At most one 1-simplex from v_i to v_j

We don't spend much time on general topology, but ...

we need to see examples...

Last time:

$(X, \mathcal{O})$ is a topological space

$\sim$ equivalence relation on X

$$X/\sim \; = \left\{ \begin{array}{c} \text{equivalence classes} \\ \text{in } X \text{ w.r.t. } \sim \end{array} \right\}$$

$U \subset X/\sim$ is open if

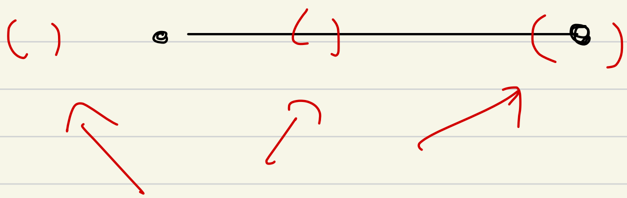
$W \subset X$ is open,

$U = W/\sim$.

Example:

$[0,1] \subset \mathbb{R}$

relatively open sets in $[0,1]$



W open in $\mathbb{R}$

$W \cap [0,1]$ is open

Say $A \subset X$, X is a top space:

$X/A = X/\sim$ where

① $\sim$ collapses all of A to one point

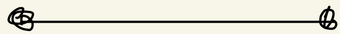
② more formally:

$$x_1 \sim x_2 \iff \begin{cases} x_1 = x_2, \text{ or} \\ x_1, x_2 \in A \end{cases}$$

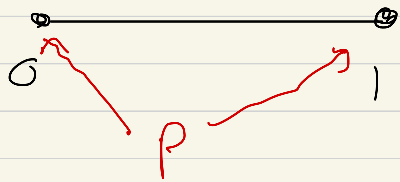
E.g. $X = [0,1]$, $A = \{0,1\}$

$X/A = [0,1]$ with $0 \sim 1$

$X \quad [0,1]$

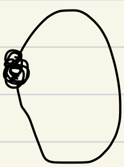


$X/\{0,1\}$



glue 0 & 1 together

$1=0$

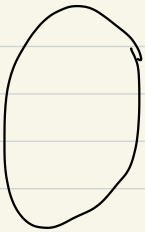


$\simeq S^1$

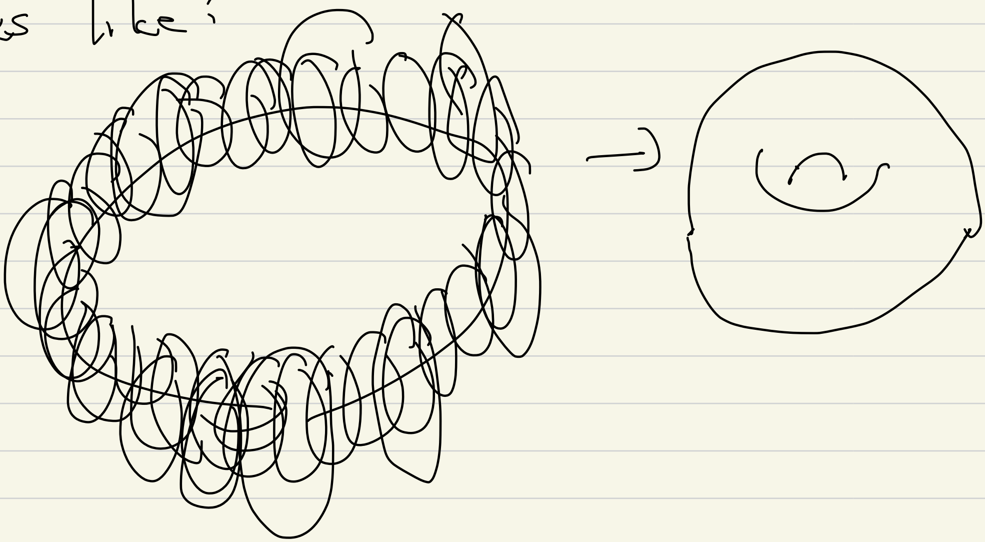
$= \{(x,y) \mid x^2+y^2=1\}$

$T = S^1 \times S^1 = T^2$

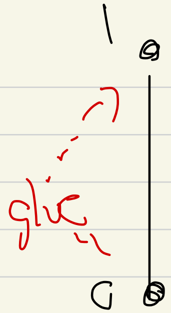
2-dim torus

S^1  $\subset \mathbb{R}^2$

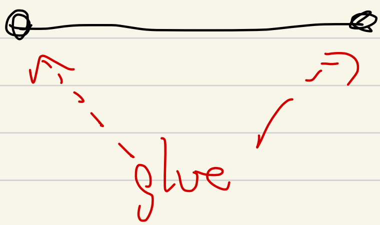
looks like:



S^1



S^1

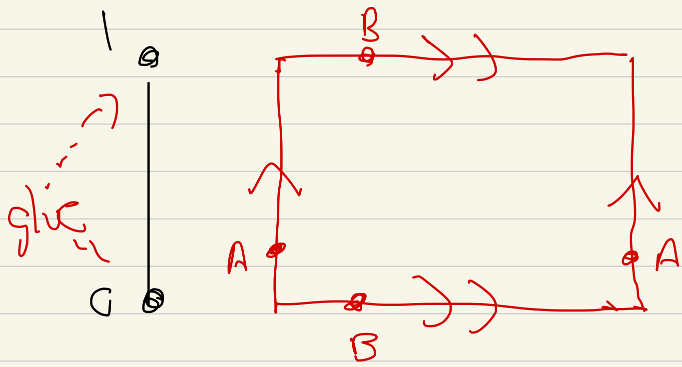


$[0,1]$

$0 \sim 1$

$T^2 = S^1 \times S^1$

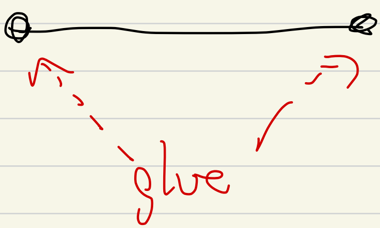
S^1



$[0,1]$

$\times [0,1]$

S^1



$[0,1]$

$0 \sim 1$

① $(X, \mathcal{O})$ is a topological space, $X' \subset X$, then X' with the subset topology (induced from $(X, \mathcal{O})$)

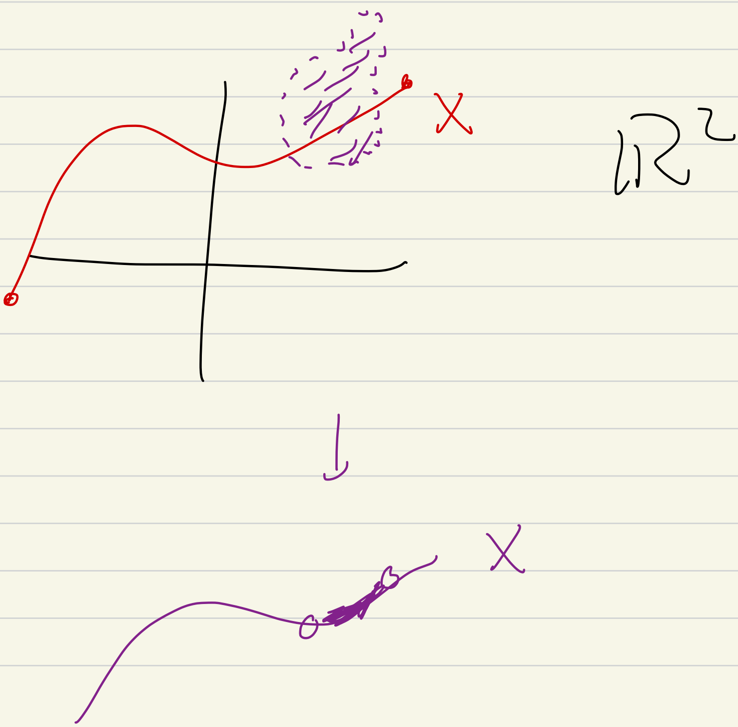
$(X', \mathcal{O}')$;

$U \in \mathcal{O}'$ (open subset of X')

$\Leftrightarrow$ $U = X' \cap W$, where W is open in X

Exercise: Prove $(X', \mathcal{O}')$ is a topological space, i.e. any finite intersection of open subsets (elements of $\mathcal{O}'$) is again open, an arbitrary union of open sets is, again, open.

E.g.,



(2) If $(X_1, \mathcal{O}_1)$, $(X_2, \mathcal{O}_2)$,

then $X_1 \times X_2$ has the

"product topology"

$(X_1 \times X_2, \mathcal{O})$

where $U \in \mathcal{O}$ (U is open set in $X_1 \times X_2$) iff

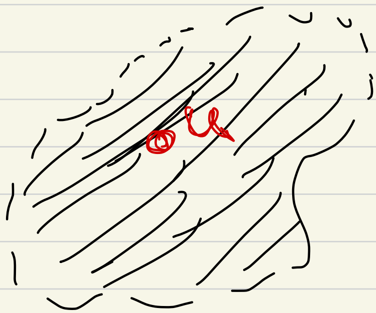
U is a union of sets $U_1 \times U_2$

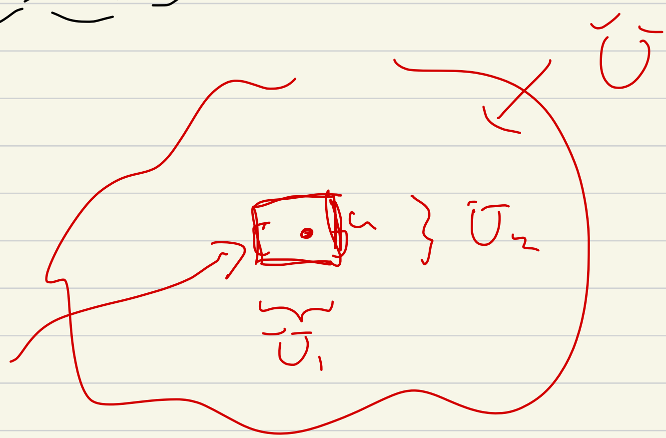
U_1, U_2 are open in X_1, X_2

$\mathbb{R}, \mathbb{R}, \qquad \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

$U \subset \mathbb{R} \times \mathbb{R}$ in the product topology iff for each $u \in U$,

$u = (x_1, x_2) \in \mathbb{R}^2$

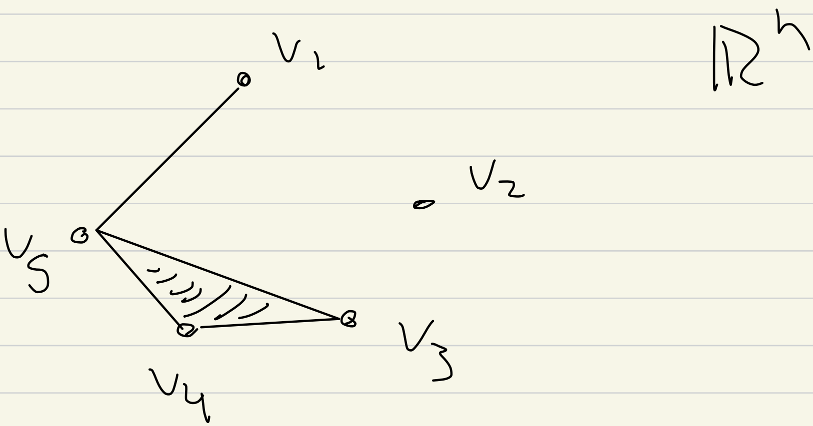




$U_1 \times U_2 \subset \tilde{U}$

Cylinder, cone, suspensions ...

Say K is a simplicial complex in $\mathbb{R}^n$



$$K = \{\emptyset, \{v_1\}, \dots \{v_5\},$$
$$\mathrm{Conv}\{v_1, v_5\}, \dots,$$
$$\mathrm{Conv}\{v_5, v_3, v_4\}, \dots \}$$

Abstract simplicial complex K^{abs}:

associated to K

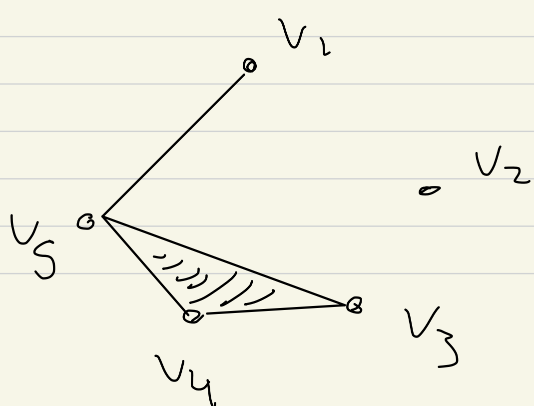
$K^{abs} = \{ \phi, \{v_1\}, \dots, \{v_5\},$

$\{v_1, v_5\}, \dots$

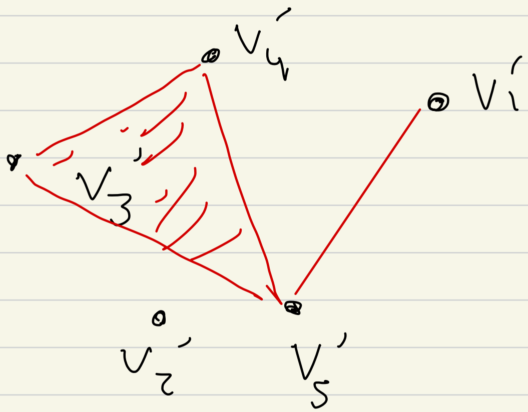
$\{v_5, v_3, v_4\} \dots \}$

$|K| = |K|_{geom} \subset \mathbb{R}^n$

$= \bigcup_{s \in K} s$



Claim: If we have another simplicial complex, K', in $\mathbb{R}^{n'}$



say that K^{abs} <u>isomorphic</u> to $(K')^{abs}$

if

$$f: \text{Vertices of } K^{abs} \xrightarrow{\text{bijection}} \text{Vertices of } (K')^{abs}$$

s.t. $S \subset$ Vertices of K^{abs}

$$S \in K^{abs} \iff f(S) \in (K')^{abs}$$

Theorem: If so $(K^{abs}$ and $(K')^{abs}$ are isomorphic, then

$$|K| \overset{\text{homeomorphic}}{\simeq} |K'|$$