CPSC 531F March 5, 2025

- Define for X top space:

$H_i^{sing}(X)$ with many properties, e.g.

① $\forall$ simp. complex K:

simplicial

$$\boxed{H_i^{sing}(|K|) \simeq H_i(K_{abs})}$$

② If $f: X \to Y$ continuous, there is

$$f_*: H_i^{sing}(X) \to H_i^{sing}(Y)$$

③ $\boxed{f, g\ X \to Y \text{ are } \underline{\text{homotopic}}}$, then

f_*, g_* agree.

- "n-simplex" means $\mathrm{Conv}(\vec{a}_0, \ldots, \vec{a}_n)$, $\vec{a}_0, \ldots, \vec{a}_n$ in general position, but

"ordered n-simplex" remembers the order of $\vec{a}_0, \ldots, \vec{a}_n$

- Define the standard n-simplex, Δ^n

- For each map $\Delta^n \to X$, we have $n+1$ <u>faces</u> $\Delta^{n-1} \to X$.

- Define $\underbrace{\mathcal{C}_i^{\mathrm{sing}}(X)}_{\text{ginormous}}$, $\underbrace{H_i^{\mathrm{sing}}(X)}_{\text{a bit mysterious}}$

Admin

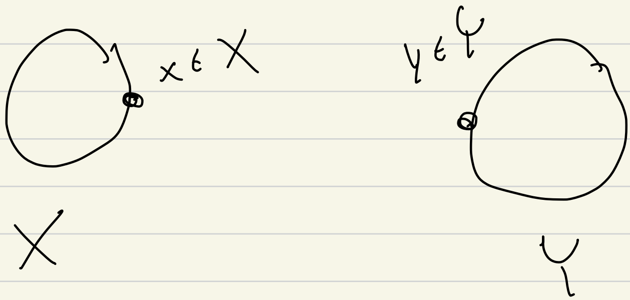
Exercises:

- You can skip exercises in §B.1 if you've seen point-set topology
- Still, you might want to do Exercise B.4 (relatively open sets in $X' \subset X$)
- Do exercises in §B.2.

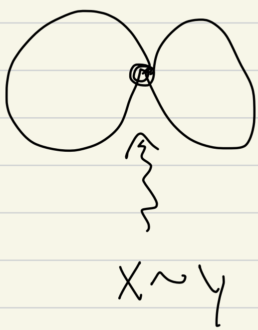
B.16 and B.17 are more subtle aspects of topology

– Exercises B.13 + B.14 are very important:

wedge sum:



$X \vee Y$:



= disjoint union of X and Y

If X, Y connected $\Rightarrow H_i(X \vee Y) = H_i(X) \oplus H_i(Y)$

Singular homology outline:

See Hatcher, Chapter 2, but our notation will be more careful:

- Ordered simplex, $\mathcal{S} \leftrightarrow (\vec{a}_0, \ldots, \vec{a}_n)$
- Standard n-simplex, Δ^n
- Ordered simplex + Δ^n

- Singular n-simplex in X:

  Continuous map: $\sigma : \Delta^n \longrightarrow X$

- The "boundary" of a singular n-simplex

$$\partial_n \sigma \overset{\text{def}}{=} \sum_{j=0}^{n} (-1)^j \, \sigma\Big|_{\vec{e}_1, \dots, \widehat{\vec{e}_j}, \dots, \vec{e}_{n+1}}$$

where

$$\sigma\Big|_{\vec{e}_1, \dots, \widehat{\vec{e}_j}, \dots, \vec{e}_n} : \Delta^{n-1} \longrightarrow X$$

The idea:

Define: singular n-simplex in X

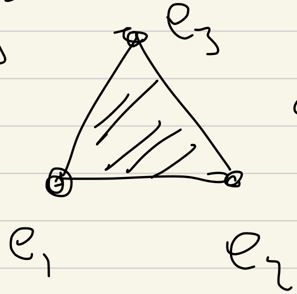
Define: ∂_n of singular n-simplex in X

Fix

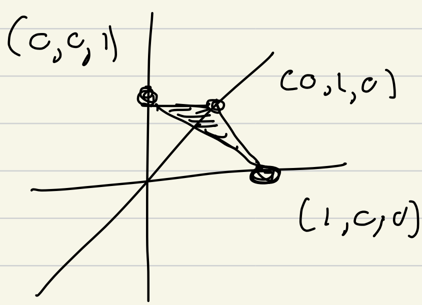
$(1,0,\dots,0)$ $\qquad$ $(0,0,\dots,0,1)$

$$\Delta^n = \mathrm{Conv}(\vec{e}_1,\dots,\vec{e}_{n+1}) \subset \mathbb{R}^{n+1}$$

Δ^2



$\subset \mathbb{R}^3$



Let $X = (X, \mathcal{O})$ be a topological space. A <u>singular</u> n-simplex is any continuous map

$$\sigma : \Delta^n \longrightarrow X .$$

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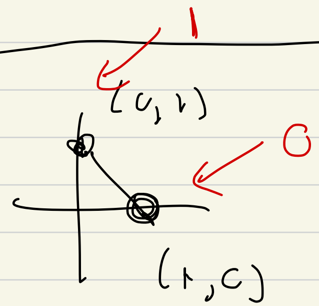
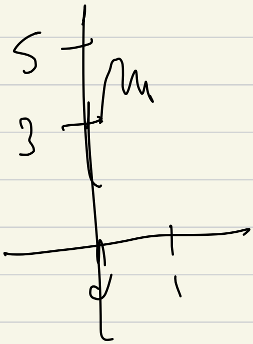
e.g. $\Delta^1 =$



$\cong [0, 1]$

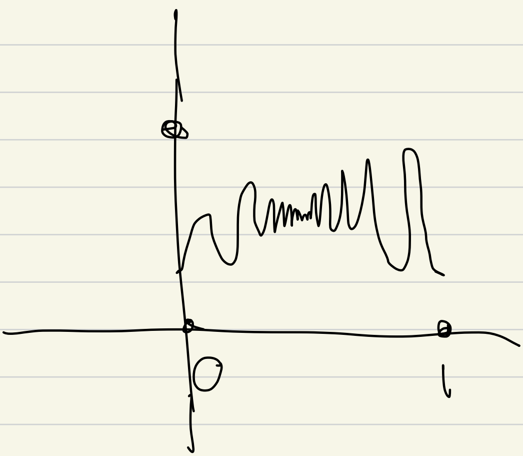
$$X = [3, 5] \subset \mathbb{R}$$

$\Delta^1 \longrightarrow X$ any continuous map



$$[0,1] \to [3,5]$$

$$(0,1] \to [0,1)$$



$$C_n^{sing}(X) = \left\{ \text{all } \mathbb{R}\text{-linear combinations of maps } \sigma : \Delta^n \to X \right\}$$

here $\sigma : \Delta^n \to X$

$\sigma' : \Delta^n \to X$

we say $\sigma = \sigma'$ in $C_n^{sing}(X)$

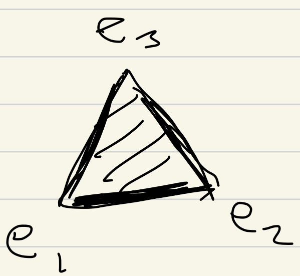
iff σ, σ' are the exact same map

$$\Delta^n = \left\{ (t_0, t_1, \dots, t_n) \;\middle|\; \begin{array}{l} t_0, \dots, t_n \\ \geq 0, \text{ real} \\ t_0 + \dots + t_n = 1 \end{array} \right\}$$

$\Delta^n \to X$

Next: define ∂_n

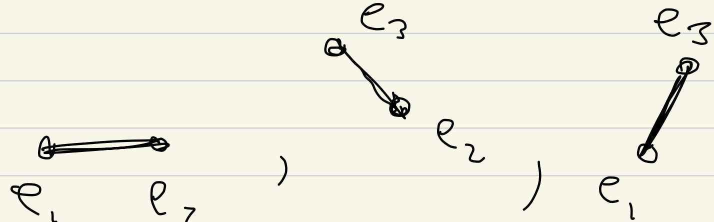
$$\partial_n : C_n^{sing}(X) \longrightarrow C_{n-1}^{sing}(X)$$



$$\Delta^2 \subset \mathbb{R}^3$$

any $\sigma : \Delta^2 \longrightarrow X$

we can restrict σ to



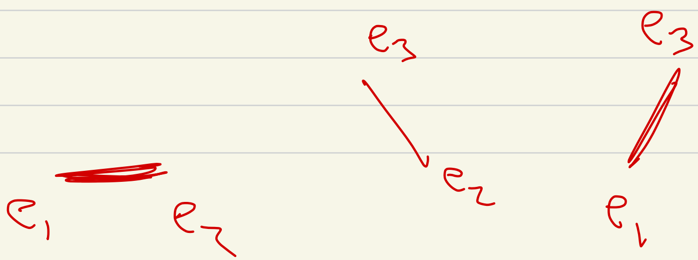
See Hatcher ...

We think of

$$\Delta^n = \mathrm{Conv}(\vec{e}_1, \dots, \vec{e}_{n+1}) < \mathbb{R}^{n+1}$$

as $\left(\text{an } n\text{-simplex} \right) + \left(\text{remember the order of the vertices} \right)$

boundary of Δ^2:



we want to get

3 maps $\Delta^1 \to X$

Define: An ordered n-simplex:

$\vec{a}_0, \dots, \vec{a}_n \in \mathbb{R}^N$ in general position, but

$$\Delta = \Big(\underbrace{\mathrm{Conv}(\vec{a}_0, \dots, \vec{a}_n)}_{\text{in } \mathbb{R}^N}, \underbrace{(\vec{a}_0, \dots, \vec{a}_n)}_{\substack{\text{listing} \\ \text{the vertices} \\ \text{in order}}} \Big)$$

If we have ordered n-simplex

$$\left(\mathrm{Conv}(\vec{a}_0, \dots, \vec{a}_n), \quad (\vec{a}_0, \dots, \vec{a}_n) \right)$$

there there's a unique map

$$\Delta^n \longrightarrow \mathrm{Conv}(\vec{a}_0, \dots, \vec{a}_n)$$

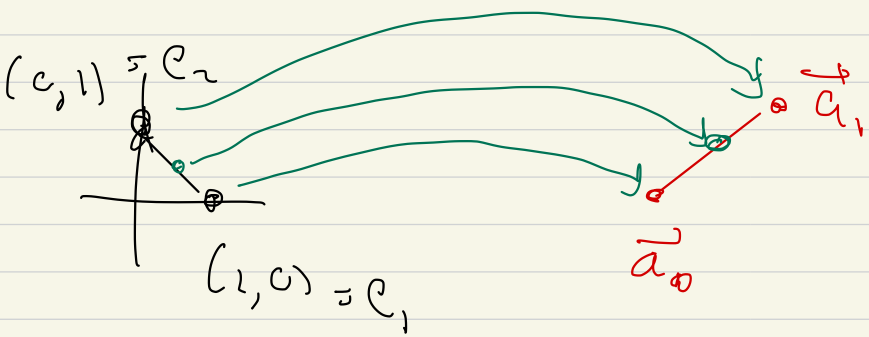
$$(t_0, \dots, t_n) \longmapsto t_0\vec{a}_0 + t_1\vec{a}_1 + \dots + t_n\vec{a}_n$$

$$(t_0, \dots, t_n \geq 0 \text{ in } \mathbb{R}, \quad t_0 + \dots + t_n = 1)$$

Can think an ordered n-simplex:

$(\vec{a}_0, \dots, \vec{a}_n)$ a sequence in $\mathbb{R}^N$, $\vec{a}_0, \dots, \vec{a}_n$ are in general position

Δ^1

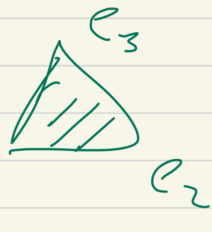


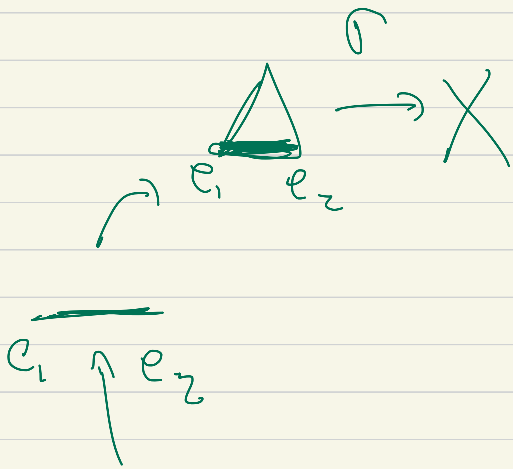
$$\begin{array}{c} (\vec{e}_1, \vec{e}_2) \\ \downarrow \quad \downarrow \\ (\vec{a}_0, \vec{a}_1) \end{array}$$

$\vec{a}_0, \vec{a}_1$ in $\mathbb{R}^N$ in general position

Now: $\sigma : \Delta^2 \longrightarrow X$

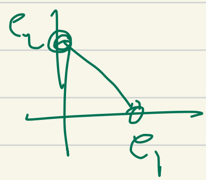
Δ^2 :





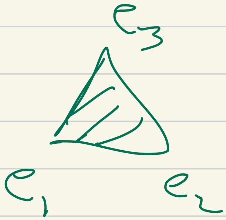
$\sigma\big|_{e_1, e_2}$

Δ^1 :



$\Delta^1 \to X$

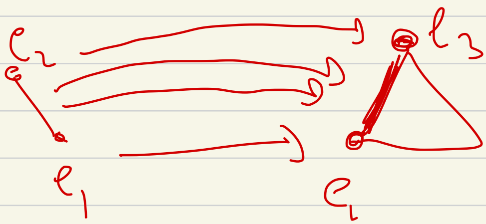
$(e_1, e_2, \hat{e}_3) = (e_1, e_2)$



$$(e_1, \hat{e_2}, e_3)$$

$$(e_1, e_3)$$

Δ^1



$$\Delta^2 \xrightarrow{\sigma} X$$

$\partial_2 \sigma$, $\sigma : \Delta^2 \to X$

$$\sum_{j=0}^{2} (-1)^j \, \sigma|_{(e_1, e_2, e_3) \text{ removing } e_j}$$

$$\sigma : \Delta^n \to X$$

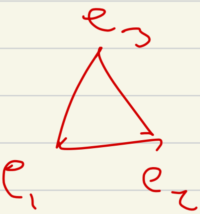
$$\partial_n \sigma \overset{\text{def}}{=} \underbrace{\sum_{j=0}^{n} (-1)^j \, \sigma\Big|_{(e_1, \ldots, \hat{e}_j, \ldots, e_n)}}_{\Delta^{n-1} \to X}$$

Then: $\partial_n : C_n^{\text{sing}}(X) \to C_{n-1}^{\text{sing}}(X)$

$$\partial_{n-1}\partial_n = 0$$

E.g.:

$$\Delta^2 \xrightarrow{\sigma} X$$

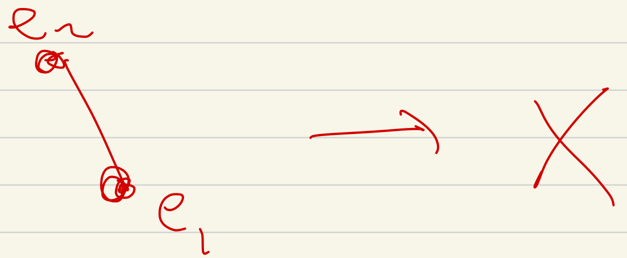


$$\xrightarrow{\sigma} X$$

$$\partial_2(\quad) = \sigma\big|_{(e_2,e_3)} - \sigma\big|_{(e_1,e_3)}$$

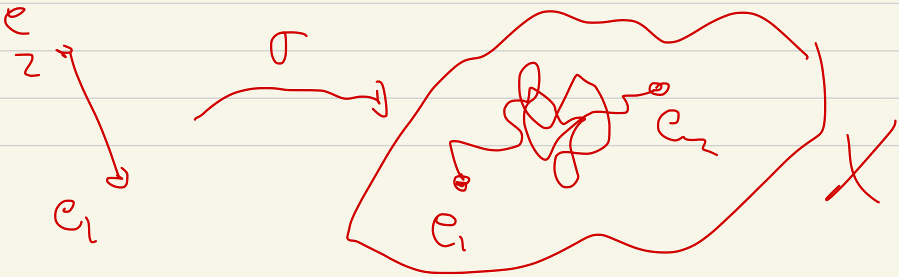
$$+ \sigma\big|_{(e_2,e_3)}$$

$\Delta^1 \longrightarrow X$



in $\mathbb{R}^2$

$$\boxed{\partial_1 \sigma := \sigma\big|_{e_2} - \sigma\big|_{e_1} = \sigma(e_2) - \sigma(e_1)}$$



$$(\partial_2 \sigma) = \left(\sigma|_{(e_2,e_3)} - \sigma|_{(e_1,e_3)} + \sigma|_{(e_1,e_2)} \right)$$

$\partial_1(\partial_2\sigma) =$ Six maps $\Delta^0 \to X$

are point

$$= \sigma(e_3) - \sigma(e_2)$$

$$- (\sigma(e_3) - \sigma(e_1))$$

$$+ (\sigma(e_2) - \sigma(e_1))$$

$$= 0$$

Def: We get a chain

$$\cdots \xrightarrow{\partial_2} C_1^{sing}(X) \xrightarrow{\partial_1} C_0^{sing}(X) \to 0$$

and $\partial_i \partial_{i+1} = 0$ for all i,

so

$$H_i^{sing}(X) \stackrel{def}{=} \ker \partial_i \big/ \operatorname{Im}(\partial_{i+1})$$

Next time: (1) $X = \{p\}$ we get the right thing