CPSC 531F March 7, 2025

Last time:

$$\Delta^n = \mathrm{Conv}(\vec{e}_1, \ldots, \vec{e}_{n+1}) \subset \mathbb{R}^{n+1}$$

---

ordered simplex: we remember the order of the vertices.

$$\mathcal{C}_n^{\text{singular}}(X) = \left\{ \begin{array}{l} \text{formal } \mathbb{R}\text{-linear} \\ \text{combinations of} \\ \text{maps } \sigma : \Delta^n \to X \end{array} \right\}$$

$$\partial_n \sigma \stackrel{\text{def}}{=} \sum_{j=1}^{n+1} (-1)^{j+1} \sigma\Big|_{(\vec{e}_1, \ldots, \hat{\vec{e}}_j, \vec{e}_{n+1})}$$

$$H_n^{\text{sing}}(X) \stackrel{\text{def}}{=} \ker \partial_n \big/ \mathrm{Image}(\partial_{n+1})$$

Why interesting:

- $H_n^{sing}(X)$ defined for any topological space

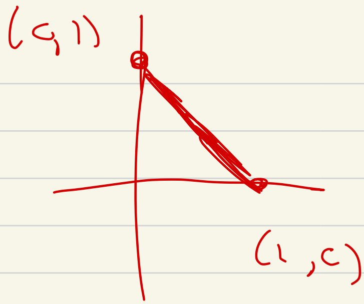
- $f: X \to Y$ gives maps $f_*: H_i(X) \to H_i(Y)$

- $H_n^{sing}(X)$ agrees with $H_n^{simp}(K^{abs})$

- f & g homotopic $\Rightarrow$ $f_* = g_*$ as maps $H_i^{sing}(X) \to H_i^{sing}(Y)$

- Application: Brouwer fixed point theorem.

Δ^1 = 1-simplex



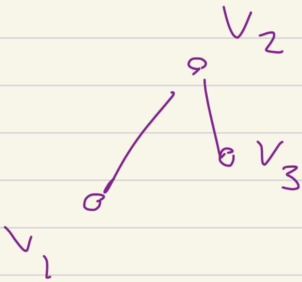
Path in X!

a map $[0,1] \longrightarrow X$

these can be very bad maps

<u>singular</u> $\longleftrightarrow$ to remind us that maps $\Delta^n \longrightarrow X$ can be wild

Simplicial homology of an abstract complex



$\mathcal{C}_1^{simp}(\quad)$

includes $3[v_1, v_2]$

$5[v_1, v_2] + 2[v_2, v_3]$

$[v_1, v_2] = -[v_2, v_1]$

$v_1 \longrightarrow v_2$ $\qquad$ $v_1 \longleftarrow v_2$

$$\cdots \to C_2^{sing}(X) \xrightarrow{\partial_2} C_1^{sing}(X) \xrightarrow{\partial_1} C_0^{sing}(X) \xrightarrow{\partial_0} 0$$

$$\partial_1 \partial_2 = 0, \qquad \partial_{n-1} \partial_n = 0$$

$$H_n^{sing}(X) \overset{def}{=} \ker \partial_n \big/ \operatorname{Image}(\partial_{n+1})$$

$\cdots$

$f : X \to Y$

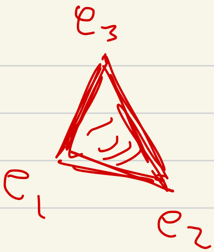
$\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$

$\sigma \rightsquigarrow f\sigma : \Delta^n \to Y$

gives

$$f_{\#,i} : \begin{tikzcd} \mathcal{C}_i^{sing}(X) \arrow[r, "f_{\#,i}"] \arrow[d, "\partial_i = \partial_i(X)"] & \mathcal{C}_i^{sing}(Y) \arrow[d, "\partial_i(Y)"] \\ \mathcal{C}_{i-1}^{sing}(X) \arrow[r, "f_{\#,i-1}"'] & \mathcal{C}_{i-1}^{sing}(Y) \end{tikzcd}$$

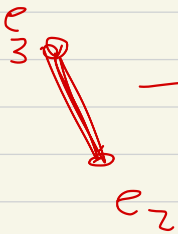
commutes



$\xrightarrow{\sigma} X \xrightarrow{f} Y$

∂_2

built from restrictions



$\xrightarrow{\sigma|_{(e_2,e_3)}} X \xrightarrow{f} Y$

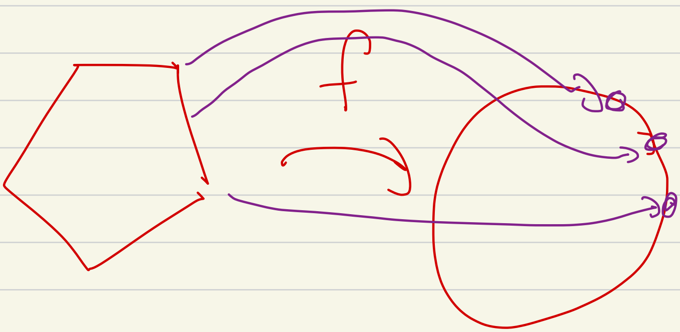
Implies

$$f_* : H_i^{sing}(X) \to H_i^{sing}(Y)$$

[Similar map for abstract simplicial complexes ...]

Say that X, Y are homeomorphic

$X \qquad\qquad Y$



pentagon (cycle length 5) $\qquad S^1$

f is a bijection, f, f^{-1} are continuous

Then:

$$f_* : H_i^{sing}(X) \to H_i^{sing}(Y)$$

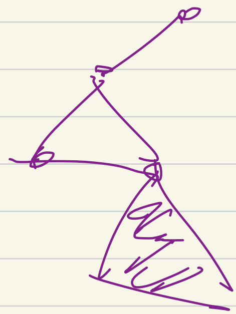
$$f^{-1}_* : H_i^{sing}(X) \leftarrow H_i^{sing}(Y)$$

$$\underbrace{f^{-1} f}_{\text{identity}} : X \xrightarrow{f} Y \xrightarrow{f^{-1}} X$$

$$\underbrace{H_i^{sing}(X) \xrightarrow{f_*} H_i^{sing}(Y) \xrightarrow{(f^{-1})_*} H_i^{sing}(X)}_{(id)_*}$$

We won't prove!

K simplicial complex



$K = \{ \text{simplices} \}$

$K^{abs} = \{ \text{subsets of } V(K) \}$

$$|K|_{geom} = \bigcup_{X \in K} X$$

$$H_i^{simp}(K^{abs}) \longrightarrow H_i^{sing}(|K|)$$

Homework!

S^0 $\quad \cdot \qquad \cdot \quad$ in $\mathbb{R}$

$\qquad\quad -1 \qquad 1$

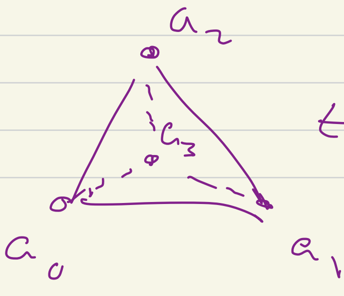
$$\beta_i(S^0) = \begin{cases} 2 & \text{if } i=0 \\ 0 & \text{if } i \geq 1 \end{cases}$$

$n \geq 1$

$S^2 \subset \mathbb{R}^3$

$$\beta_i(S^n) = \begin{cases} 1 & i=0,n \\ 0 & i \neq 0,n \end{cases}$$

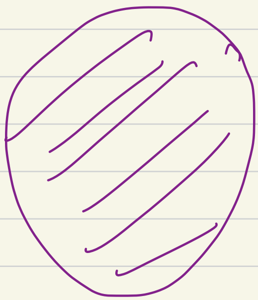
$S^2 =$



$\longleftrightarrow$ $\mathrm{Power}\{a_0,a_1,a_2,a_3\} \setminus \{\{a_0,a_1,a_2,a_3\}\}$

$\mathbb{D}^n = \{ \vec{x} \in \mathbb{R}^n \mid |\vec{x}| \leq 1 \}$

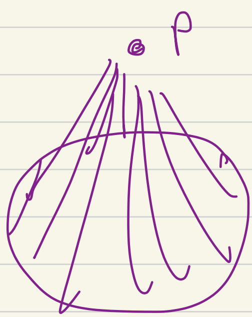
$x_1^2 + \dots + x_n^2 \leq 1$



$\mathbb{D}^2 \subset \mathbb{R}^2$

boundary of $\mathbb{D}^2$ is S^1

$\mathbb{D}^2$: S^1



$\mathbb{D}^2 = \mathrm{Cone}_p(S^1)$

$$B_i(\mathbb{D}^n) = \begin{cases} 1 & i = 0 \\ 0 & i > 0 \end{cases}$$

$=$

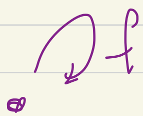
Thm: (Brouwer fixed point theorem)

If $n \geq 0$, $f : \mathbb{D}^n \to \mathbb{D}^n$ (continuous)

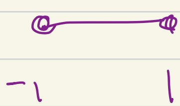
then f has a fixed point,

i.e. $\vec{x} \in \mathbb{D}^n$ s.t. $\vec{f}(\vec{x}) = \vec{x}$.

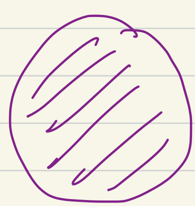
$\mathbb{D}^0$

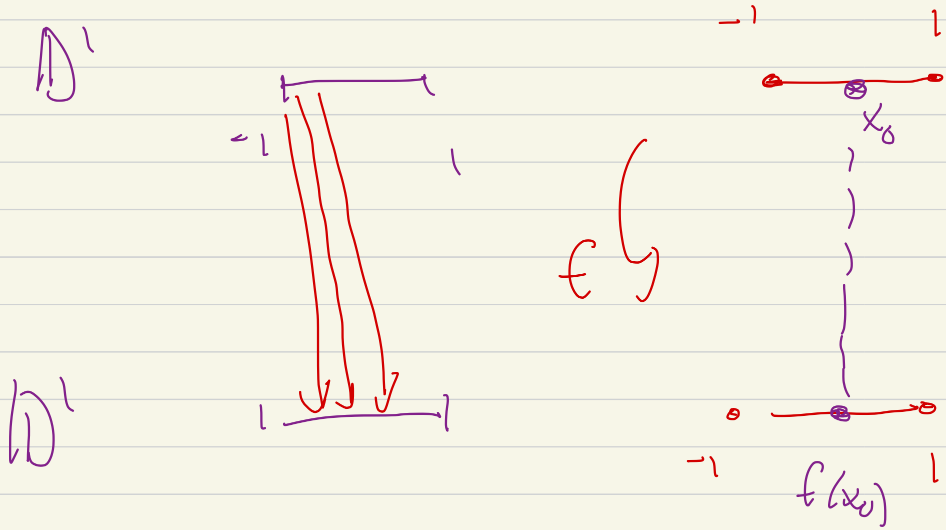


$\mathbb{D}^1$



$\mathbb{D}^2$





Given $f: [-1,1] \to [-1,1]$

for any $x \in [-1,1]$

either (1) $f(x) \geq x$ $\leftarrow -1$

if no fixed point

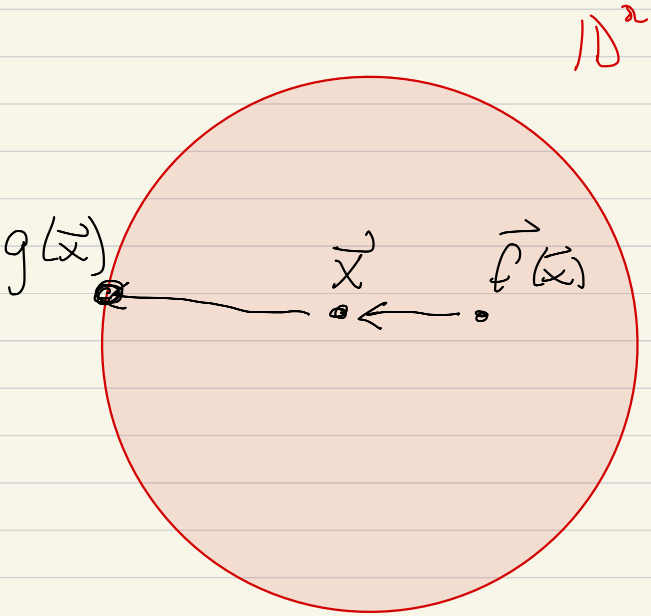
or

(2) $f(x) \leq x$ $\leftarrow 1$

Proof: Say $f: \mathbb{D}^n \to \mathbb{D}^n$

s.t. $f(x) \neq x$ for all $x \in \mathbb{D}^n$.

Define



Claim:

$$\left\{ \vec{x} + \alpha(\vec{x} - \vec{f}(x)) \;\middle|\; \alpha \geq 0 \right\}$$

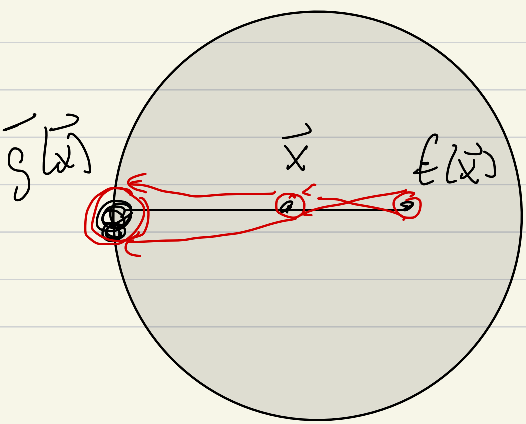
hits

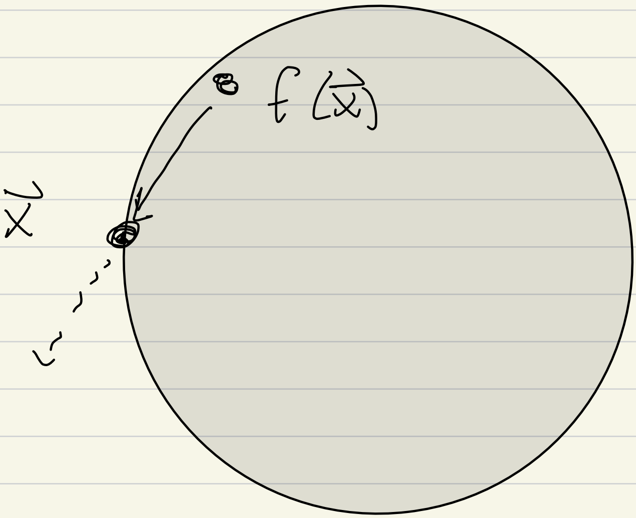
S^{n-1} at one point

$$g(\vec{x}) = S^{n-1} \cap \{ \vec{x} + \alpha(\vec{x} - f(\vec{x})) \text{ s.t. } \alpha \geq 0 \}$$

Claim:

$g(\vec{x})$ is continuous





So!

$$g(\vec{x}) : \mathbb{D}^n \longrightarrow S^{n-1}$$

and

$$g|_{S^{n-1}} = \text{identity map}$$

Now

$$\begin{tikzcd} S^{n-1} \arrow[r, hook, "\text{subset}"] \arrow[rr, bend right, "id_{S^{n-1}}"'] & D^n \arrow[r, "g"] & S^{n-1} \end{tikzcd}$$

- - -

contradiction !