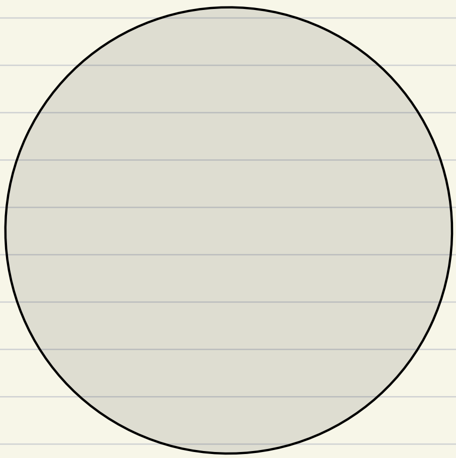
CPSC 531F March 10, 2025

- Today:
  - Finish Brouwer Fixed Point Theorem
  - Perron-Frobenius theorem

    Special case: stationary distribution of a Markov chain
  - Nash Equilibrium, prove existence

Theorem: Let

$$\mathbb{D}^n = \{ \vec{x} \in \mathbb{R}^n \mid |\vec{x}| \leq 1 \}$$



$\subset \mathbb{R}^n$

$$\partial \mathbb{D}^n : \overline{\mathbb{D}^n} \setminus \text{interior}(\mathbb{D}^n)$$

$X \subset \mathbb{R}^n$ (subset)

$$\text{interior}(X) = \bigcup_{\substack{\text{open sets} \\ W \subset X}} W$$

interior(X) = the largest open subset of X in $\mathbb{R}^n$

$$\overline{X} = \mathbb{R}^n \setminus \left(\text{interior}\left(\mathbb{R}^n \setminus X\right)\right)$$

= closed set

$$\partial X = \overline{X} \setminus \text{interior}(X)$$

$$\partial \mathbb{D}^n =$$

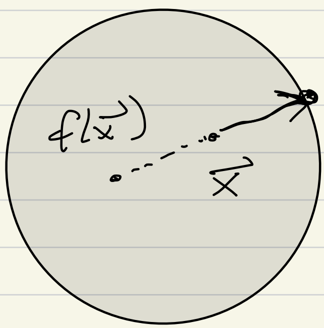
$$S^{n-1} = \left\{ \vec{x} \in \mathbb{R}^n \,\middle|\, |\vec{x}| = 1 \right\}$$

Thm [Brouwer fixed point theorem]

If $f : \mathbb{D}^n \to \mathbb{D}^n$, then f

has a fixed point.

Pf:



$\mathbb{D}^n$ is convex

$\vec{x} \neq f(\vec{x})$

$\mathrm{Ray}(\vec{x}, \vec{x} - f(\vec{x}))$

$$\{ \vec{x} + t(\vec{x} - f(\vec{x})) \mid t \geq 0 \}$$

meets $\mathbb{S}^{n-1}$ in one place

$$g(\vec{x}) = \mathrm{Ray}(\vec{x}, \vec{x} - \vec{f}(\vec{x})) \cap S^{n-1}$$

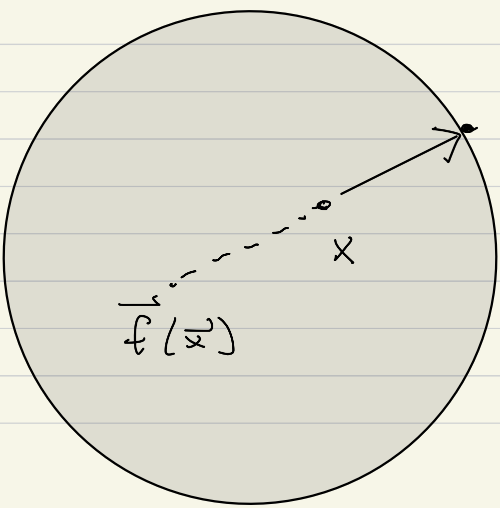
Claim:

(1) $g : \mathbb{D}^n \longrightarrow S^{n-1}$

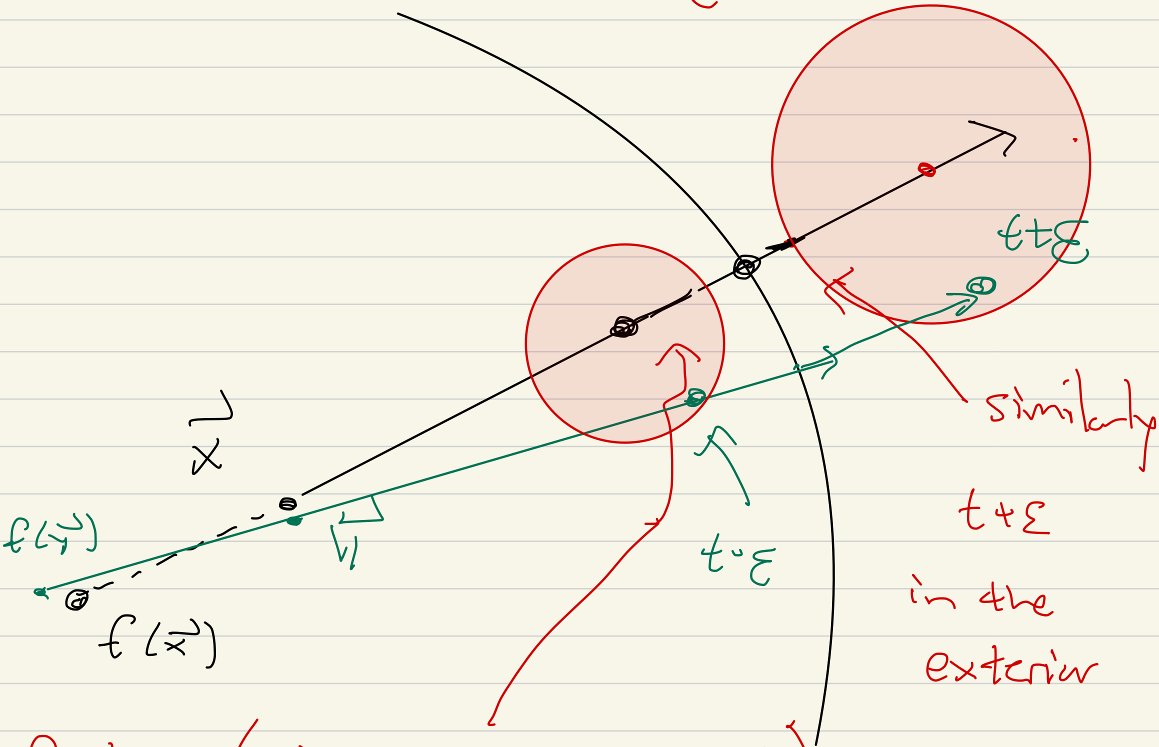
(2) $g(\vec{s}) = \vec{s}$ if $\vec{s} \in S^{n-1}$

(3) g is continuous

$\leftarrow \vec{x} + t(\vec{x} - \vec{f}(\vec{x}))$



Fix $\vec{x}$, t as such, for all $\varepsilon > 0$

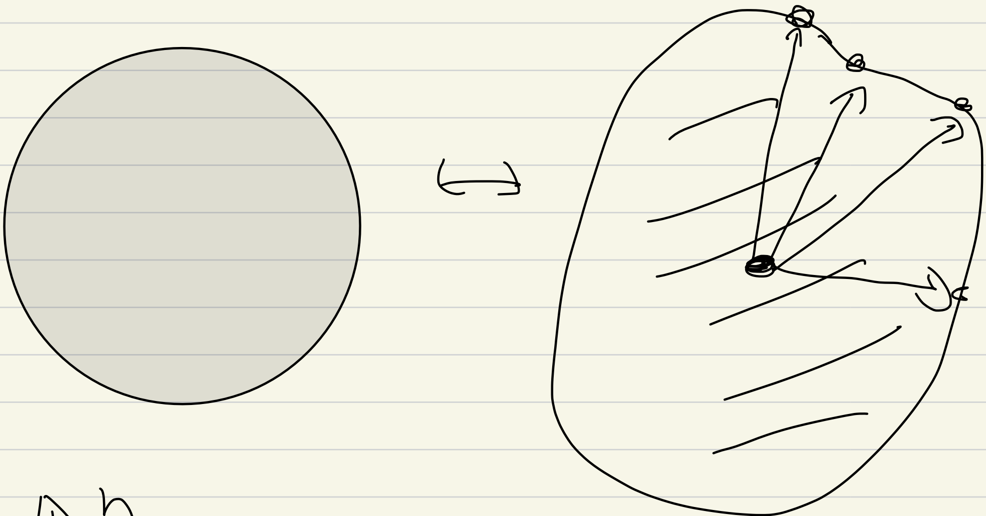


Ball $\left(\vec{x} + (t-\varepsilon)(\vec{x} - f(\vec{x}))\right)$

radius $\delta > 0$

in interior

Remark: homeomorphic

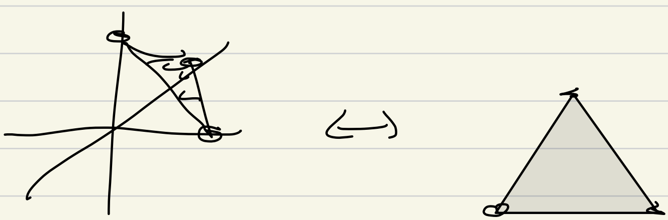


$\mathbb{D}^n$

X closed

homeo bounded in $\mathbb{R}^n$

Δ^n



But same argument when

$X \overset{\text{homeomorphic to}}{\cong} \mathbb{D}^n$

but X is not convex, falls apart.

Now knowing (1) - (3) we get a contradiction ...

$$S^{n-1} \xrightarrow{\iota} \mathbb{D}^n \xrightarrow{g} S^{n-1}$$

$\uparrow$ inclusion

$$H_i(S^{n-1}) \xrightarrow{\iota_*} H_i(\mathbb{D}^n) \xrightarrow{g_*} H_i(S^{n-1})$$

If $X \xrightarrow{f} Y$

f continuous !

$\sigma : \Delta^n \longrightarrow X$ cont.

$$\begin{tikzcd} \Delta^n \arrow[r, "\sigma"] \arrow[dr, "f\sigma"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

$f\sigma : \Delta^n \longrightarrow Y$

$$f_{\#,i} : C_i^{sing}(X) \longrightarrow C_i^{sing}(Y)$$

get

$$f_{*,i} : H_i^{sing}(X) \longrightarrow H_i^{sing}(Y)$$

And if $X = Y$, $f = \text{identity}_X$

then $f_\#, f_* = \text{identity}$

$=$

Since $g\iota = \text{identity}_{S^{n-1}}$,

(*)

$$\begin{tikzcd} H_i(S^{n-1}) \arrow[r] \arrow[rr, bend right, dashed] & H_i(D^n) \arrow[r] & H_i(S^{n-1}) \end{tikzcd}$$

$g_*\iota_* = \text{identity}$

But: S^{n-1}

$n \geq 2$

$$H_{n-1}(S^{n-1}) \cong \mathbb{R}$$

but

$$H_{n-1}(D^n) = 0$$

impossible

$$(*) \Rightarrow \begin{tikzcd} \mathbb{R} \arrow[r] \arrow[rr, bend left, "0"] \arrow[rr, bend right, dashed, "\text{identity}"'] & 0 \arrow[r] & \mathbb{R} \end{tikzcd}$$

$n = 1$, similarly

$$S^{n-1} = S^0 = \bullet \qquad \bullet$$

$$H_0(S^{n-1}) \cong \mathbb{R}^2$$

$$H_0(D^n) \cong \mathbb{R}$$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r] \arrow[rr, bend right] & \mathbb{R} \arrow[r] & \mathbb{R}^2 \end{tikzcd}$$

can't be the identity

<u>Thm</u>: If $X = (X, \mathcal{O})$ is a topolog. space, $A \subset X$ is a subset we say that A is a <u>retraction</u> of X if there is a map

$$f : X \longrightarrow A$$

- continuous
- $\forall \vec{a} \in A, \ f(\vec{a}) = \vec{a}$.

has to be the identity

If so

$$\begin{tikzcd} H_i(A) \arrow[r, hook, "\text{include}"] \arrow[rr, dashed, bend left] & H_i(X) \arrow[r, "f_*"] & H_i(A) \end{tikzcd}$$

If so, then

$$\dim(H_i(A)) \leq \dim(H_i(X))$$

$$\begin{array}{ccc} \| & & \| \\ \beta_i(A) & \leq & \beta_i(X) \end{array}$$

Applications:

We say that M, $n \times n$ matrix, with non-negative entries, is a <u>Markov matrix</u> if:

$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & & & \\ \vdots & & & \\ m_{n1} & \cdots & & m_{nn} \end{bmatrix}$$

Intuitively

m_{ij} = probability of moving from i to j

row of M

$$[m_{21}\ m_{22}\ m_{23}\ \cdots\ m_{2n}]$$

$$m_{21} + m_{22} + \cdots + m_{2n} = 1$$

state: $\vec{s} = [s_1, s_2, \ldots, s_n]$

stochastic: $s_1, \ldots, s_n \geq 0$,

$s_1 + \cdots + s_n = 1$

Then

$$\vec{s}M \quad \text{also stochastic}$$

A stochastic $\vec{\pi} = [\pi_1, \ldots, \pi_n]$ is stationary for M if

$$\vec{\pi} M = \vec{\pi}$$

Thm: If ......, then $\vec{\pi}$ exists and is unique.