CPSC 531F March 12, 2025

Brouwer fixed point theorem:

$\vec{f} : \mathbb{D}^n \to \mathbb{D}^n$ (is continuous)

then $\vec{f}$ has a fixed point, i.e.

for some $\vec{x} \in \mathbb{D}^n$, $\vec{f}(\vec{x}) = \vec{x}$.

<u>Applications:</u>

(1) Perron-Frobenius theorem

(2) Markov matrices + stationary vector

(3) Existence of a Nash Equilibrium

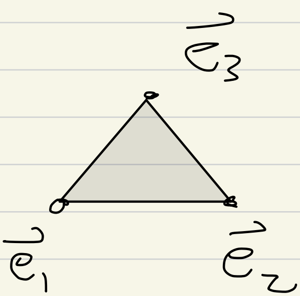
Recall:

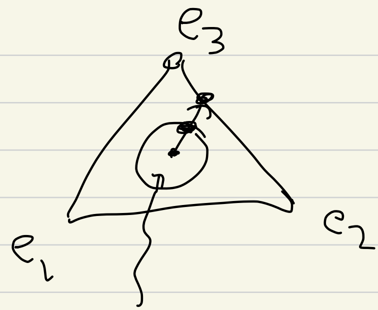
$$\Delta^{n-1} = \left\{ (x_1,\dots,x_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} x_1,\dots,x_n \geq 0, \\ x_1 + \dots + x_n = 1 \end{array} \right\}$$

Fact:

(1) $\Delta^{n-1} \overset{\text{homeomorphic}}{\cong} \mathbb{D}^{n-1}$.

Why? Idea

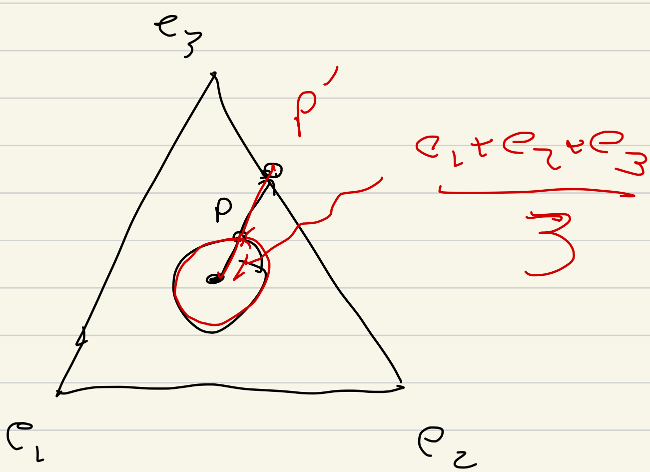
$\Delta^2 =$  in $\mathbb{R}^3$



For each

$p \in$

$$\{ x_1^2 + x_2^2 \le c^2 \} = c\, \mathbb{D}^2$$



there's a unique $p' \in \partial \Delta^2$

s.t. ray from $\frac{e_1+e_2+e_3}{3}$ to p

hits $p' \in \partial \Delta^2$

$p \mapsto p'$ sets up homeomorphism

$\mathbb{D}^2 \xrightarrow{\sim} c\mathbb{D}^2 \xrightarrow{\sim} \Delta^2$

So there is a bijection

$f : \mathbb{D}^k \xrightarrow{\sim} \Delta^k$

s.t. f, f^{-1} are continuous

and bijections.

This implies:

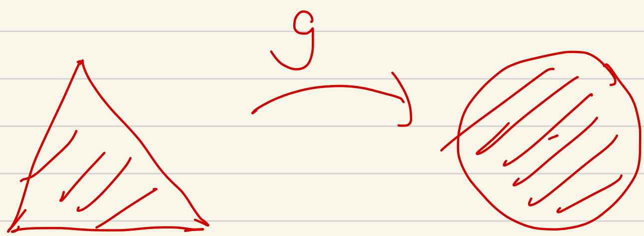
TRICK #1:

Brouwer fixed point theorem:

$\vec{f} : \Delta^n \to \Delta^n$ (is continuous)

then $\vec{f}$ has a fixed point, i.e.

for some $\vec{x} \in \Delta^n$, $\vec{f}(\vec{x}) = \vec{x}$.



fixed point $\vec{x}$

fixed point $p = g^{-1}(\vec{x})$

$$\begin{tikzcd}
\Delta^n \arrow[r, "g"] \arrow[d, "f"'] & D^n \arrow[l, bend left, "g^{-1}"] \arrow[d, "g f g^{-1}"] \\
\Delta^n \arrow[r, "g"] & D^n
\end{tikzcd}$$

Then

$$gfg^{-1}(\vec{x}) = \vec{x}$$

$$(fg^{-1})(\vec{x}) = g^{-1}(\vec{x})$$

$$f(p) = p,$$

where $p = g^{-1}(\vec{x})$

TRICK # 2

$\vec{x} \in \Delta^{n-1} \iff$ $\vec{x} = (x_1, \ldots, x_n)$

$x_i \geq 0, \quad x_1 + \ldots + x_n = 1$

$\vec{x} \in \mathbb{R}^n$ is <u>stochastic</u> if $\vec{x} \in \Delta^{n-1}$

Say we have a vector

$\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n,$

$v_1, \ldots, v_n \geq 0$ and $\vec{v} \neq \vec{0}$.

<u>Define</u>

$$\boxed{\text{Stochastic}(\vec{v}) \overset{\text{def}}{=} \frac{\vec{v}}{v_1 + \ldots + v_n}}$$

Perron-Frobenius Theorem.

Let $M \in (\mathbb{R}_{\geq 0})^{n \times n}$, i.e.

M is $n \times n$ matrix, whose entries are non-negative reals.

Say also that:

$$\forall \vec{x} \in \Delta^{n-1}, \qquad M\vec{x} \neq \vec{0}.$$

so $M\vec{x}$ has non-neg components

Then:

$$f(\vec{x}) \overset{\text{define}}{=} \text{Stochastic}(M\vec{x})$$

Then $f : \Delta^{n-1} \to \Delta^{n-1}$, so

$$f(\vec{x}) = \vec{x} \quad \text{for some } \vec{x} \in \Delta^{n-1}$$

$$\|$$

$$\text{Stochastic}(M\vec{x})$$

$$\|$$

$$\frac{M\vec{x}}{\text{sum of components of } M\vec{x}} \leftarrow \text{call this } \lambda,$$

$\lambda > 0$

$$\frac{M\vec{x}}{\lambda} = \vec{x}, \quad M\vec{x} = \lambda\vec{x}$$

~~So~~ $\vec{x}$ is an eigenvector of M, eigenvalue λ.

$$\boxed{M\vec{x} \neq \vec{0} \quad \text{for} \quad \vec{x} \in \Delta^{n-1}}$$

$n = 2$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix}$$

$x_1, x_2 \geq 0$

~~has~~ has ≥ 0 components

$$\begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{bad...}$$

Since $M \in \mathbb{R}^{n \times n}_{\geq 0}$,

$\vec{x}$ has ≥ 0 components $\implies$

$M\vec{x}$ " " "

Claim! Say that each column of M has a non-zero entry

$$\begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & \\ \vdots & & & \\ m_{n1} & & & \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ \\ \\ \\ \\ \end{bmatrix}}_{\vec{x}}$$

non-zero

0

if one is non-zero

$$M \begin{bmatrix} x_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} m_{11} \\ \vdots \\ m_{n1} \end{bmatrix}}_{\text{first col of } M} x_1$$

$\neq 0$ if some entry of col 1 of M is non-zero

So

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \checkmark$$

$$= \begin{bmatrix} .3 & 0 & 0 \\ 0 & 0 & .2 \\ 0 & .6 & 0 \end{bmatrix} \checkmark$$

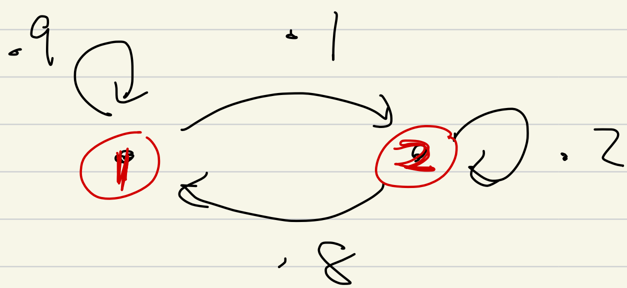
Thm: If $M \in \mathbb{R}^{n\times n}_{\geq 0}$ and in each column of M there is a positive entry, then

$$M\vec{x} \neq 0 \quad \text{if} \quad \vec{x} \in \Delta^{n-1}$$

So for some $\lambda > 0$, some $\vec{x} \in \Delta^{n-1}$ has

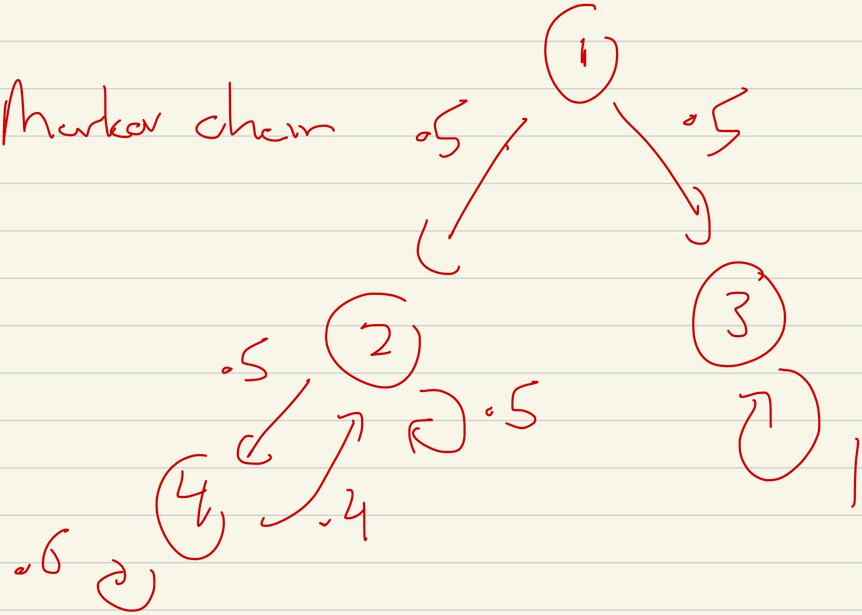
$$M\vec{x} = \lambda \vec{x}.$$

Markov chain



States: $1, \ldots, n$

Bad Markov chain



We say $M \in \mathbb{R}^{n \times n}_{\geq 0}$ is irreducible if for any $i, j \in [n]$ there are $\ell_1, \ldots, \ell_{k-1}$ s.t. setting $\ell_0 = i$, $\ell_k = j$

$$M_{\ell_0 \ell_1} > 0, \quad M_{\ell_1 \ell_2} > 0, \ldots, M_{\ell_{k-1} \ell_k} > 0$$

$$\ell_0 = i \xrightarrow{M_{\ell_0 \ell_1} > 0} \ell_1 \longrightarrow \cdots \longrightarrow \ell_k = j$$

$$\Longleftrightarrow$$

$$(M^k)_{ij} > 0 .$$

Thm: Say that $M \in \mathbb{R}^{n \times n}_{\geq 0}$ that is irreducible. Then

if $M\vec{x} = \lambda \vec{x}$ for $\vec{x} \in \Delta^{n-1}$

then

① λ is unique

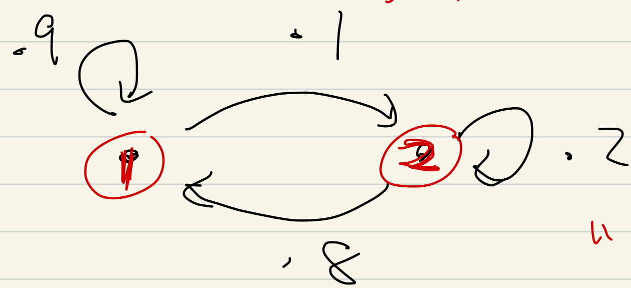
② If $M\vec{y} = \mu\vec{y}$ and $\vec{y} \neq 0$, then $|\mu| \leq \lambda$.

Rem: In Markov chains:

$P = (p_{ij})$ where each row of P is stochastic and P operates on row vectors

$$\vec{x}^T \longmapsto \vec{x}^T P$$

$P_{1\to 1} = .9$ $\qquad$ $.1 = P_{1\to 2}$



$.8 = P_{2\to 1}$ $\qquad$ $.2 = P_{2\to 2}$

$$P = \begin{bmatrix} P_{1\to 1} & P_{1\to 2} \\ P_{2\to 1} & P_{2\to 2} \end{bmatrix}$$

$$= \begin{bmatrix} .9 & .1 \\ .8 & .2 \end{bmatrix}$$

acts on ... row vectors ...