CPSC 531F March 14, 2025

(1) What does the Perron-Frobenius theorem really tell us?

(2) Start Nash Equilibrium

$G = (V, E)$ directed graph!

$A_G = \mathrm{Adj}_G$: a_{ij} = # edges from i to j

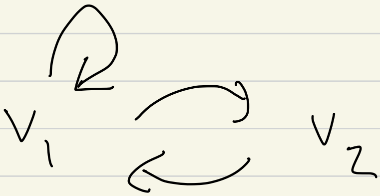
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & & \cdots & a_{nn} \end{bmatrix}$$

$$B \cdot C = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}c_{11} + b_{12}c_{21} & \cdot \\ b_{21}c_{11} + b_{22}c_{21} & \cdot \end{bmatrix}$$

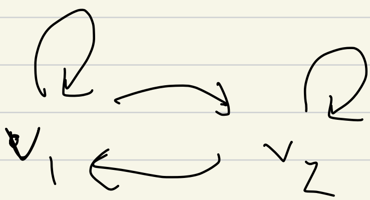
$$(B \cdot C)_{ij} = \sum_{\ell} b_{i\ell}\, c_{\ell j}$$

Fibonacci graph:



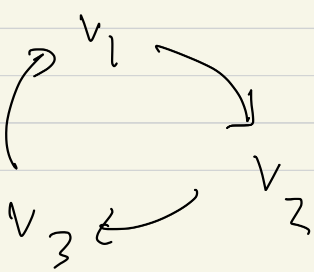
$$A = \begin{array}{c c} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

Complete directed graph



$$A = \begin{array}{c c} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

Directed Cycle



$$A = \begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} \end{array}$$

0, 1, 1, 2, 3, 5, 8, 13, ...

Fibonacci

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \qquad A^4 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, \ldots$$

$$(A^2)_{ij} = \sum_{\ell} \underbrace{(A_{i\ell})}\underbrace{(A_{\ell j})}$$

$$i \xrightarrow{A_{i\ell}} \ell \xrightarrow{A_{\ell j}} j$$

$A_{i\ell} A_{\ell j}$ = # paths i to j thru ℓ

$$(A^2)_{ij} = \#\text{ walks length 2 from } i \text{ to } j$$

$$i \circ \longrightarrow \underset{\ell \text{ any}}{\circ} \longrightarrow j$$

$$A^4 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} =$$

\# walks length 4 from 1 to 1 $= 5$

" " " " " 1 to 2 $= 3$

$\vdots$

$$A^4 = \begin{bmatrix} f_5 & f_4 \\ f_4 & f_3 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

$f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, $f_5 = 5$

$$A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$$

what is

$$\lim_{n \to \infty} \left((A^n)_{ij} \right)^{1/n}$$

= growth rate of graph

= capacity of graph

Thm: Say A has ≥ 0 entries, A is $n \times n$, A^k, for some $k \in \mathbb{N}$, has all positive entries

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Rem: $A^4 = (A^2)(A^2) = (A)(A^3)$

$A^6 = (A^3)(A^3)$

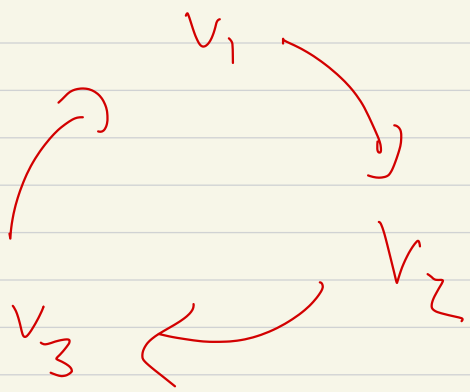
$\Longrightarrow$

walks length (exactly 6) (i,j)

$= \sum_{\ell} \Big[(\#$ walks i to ℓ length $3)$

· (# walks i to j length 3)]]

Directed cycle



$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Directed Path

$$v_1 \longrightarrow v_2 \longrightarrow v_3$$

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

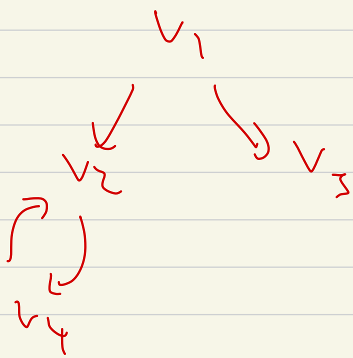
(Jordan block, nilpotent!

$$A^{\text{high enough power}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So! if one row of A is all 0's ...

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 7 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \\ \end{bmatrix}$$



Bad

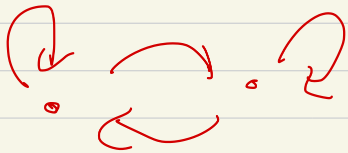
A irreducible, if $\forall\, i,j \in [n]$ we have $(A^k)_{ij} \neq 0$ for some $k \in \mathbb{N}$

Directed Path

$$v_1 \to v_2 \to v_3$$

$$A = \begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \end{array}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$



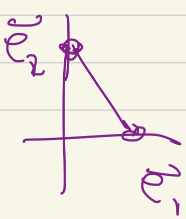
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \underbrace{\lambda = 2}$$

$$A\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \lambda\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

Perron-Frob eigenvector

in Δ^1



Perron-Frob eigenvalue

$$A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}, \ldots$$

$\mathrm{Tr}(A)$ = sum of diag elements

$$\mathrm{Tr}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2$$

here: $\lambda_1 = 2,\ \lambda_2 = 0$

<u>Thm:</u> $\mathrm{Tr}(A) = \sum_{i=1}^{n} \lambda_i$

$$\mathrm{Tr}(A^k) = \sum_{i=1}^{n} \lambda_i^k$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} ? \\ ? \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix}$$

$$\nearrow$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad ! \quad \lambda_1 = 3, \quad \vec{v}_1 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\lambda_2 = \lambda_3 = 0 \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y+z \\ '' \\ '' \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$\uparrow$$

rank 1,

$\lambda_1 = 3$ Perron-Frob

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 3A$$

$$A^3 = 9A, \quad A^m = 3^{m-1} A$$

$$\lim_{m\to\infty} \left((A^m)_{ij} \right)^{1/m} = \left(3^{m-1} \right)^{1/m} \xrightarrow[m\to\infty]{} 3$$

If $A^k > 0$ some k

and A is $n \times n$, non-neg entries

for any i, j

$$\lim_{m \to \infty} \left(\left(A^m\right)_{ij} \right)^{1/m} = \lambda_{\text{Perron-Frob}}$$

$$\lim_{m \to \infty} \left(\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^m \right)_{ij} \right)^{1/m} = \lambda_{P\cancel{e}f}$$

$$= \frac{1+\sqrt{5}}{2}$$

λ's of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$= \underbrace{\frac{1+\sqrt{5}}{2}}_{> 1.618}, \quad \underbrace{\frac{1-\sqrt{5}}{2}}_{-0.618}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = M \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} M^{-1}$$

Markov Matrices

$$
\begin{tikzcd}
v_1 \arrow[loop above, ".7"] \arrow[r, bend left, ".3"] & v_2 \arrow[l, bend left, ".6"] \arrow[loop right, ".4"]
\end{tikzcd}
$$

$$
P = \begin{bmatrix} p_{1\to1} & p_{1\to2} \\ p_{2\to1} & p_{2\to2} \end{bmatrix}
$$

Nash Equilibrium:

Play a game

Option 1 pays $a \in \mathbb{R}$

Option 2 pays $b \in \mathbb{R}$

You choose $p_1 \in [0,1]$, $p_2 = 1 - p_1$

$\text{Reward}(p_1, p_2) = a p_1 + b p_2$.

"Reward To Switch"