CPSC 531F March 17, 2025

- Shannon's capacity
  (a Perron-Frobenius eigenvalue)
- Nash Equilibrium

Both consequences Brouwer fixed pt thm!

$$f : \Delta^{n-1} \to \Delta^{n-1} = \left\{ \vec{x} \in \mathbb{R}^n, \ \vec{x} \text{ is stochastic} \right\}$$

continuous then $\vec{f}(\vec{x}) = \vec{x}$ for some $\vec{x} \in \Delta^{n-1}$

If $\vec{x} \geq \vec{0}$, $\vec{x} \neq \vec{0}$

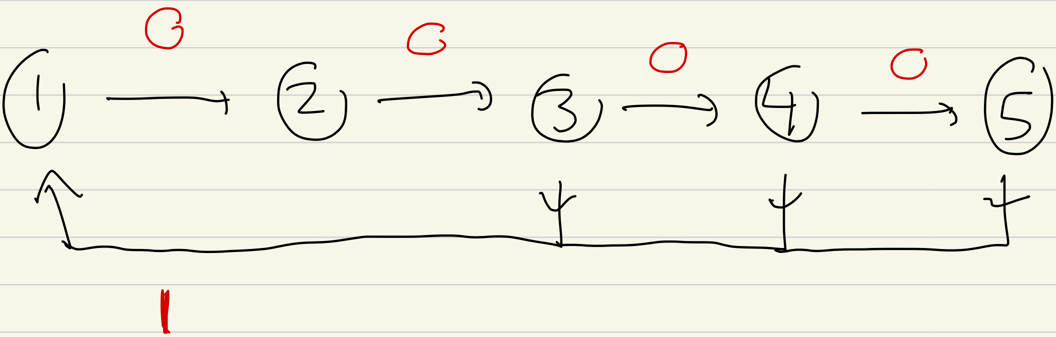
Stochastically Scale $(\vec{x})$

$$= \vec{x} \, \frac{1}{x_1 + \dots + x_n}$$

$\left(\vec{x} = (x_1, \dots, x_n) \right)$

always in Δ^{n-1}

Look at this directed graph!



Walk

$2 \to 3 \to 1 \to 2 \to 3 \to 4 \to 1$

$2 \xrightarrow{0} 3 \xrightarrow{1} 1 \xrightarrow{0} 2 \xrightarrow{0} 3 \xrightarrow{0} 4 \xrightarrow{1} 1$

0 1 0 0 0 1

- Between any two successive 1's, we see 2 to 4 0s

see

$$\cdots 1 \; 0 \; 0 \underbrace{\quad\quad}_{\text{maybe}} 1$$

(1)

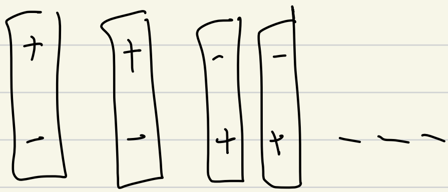
maybe
- nothing
- 0
- 00

never more than 4 successive 0's.

"(2, 4) run length limited code"

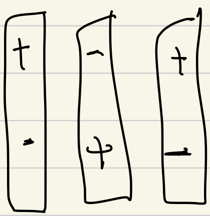
Magnetic storage (tapes)

Idea:



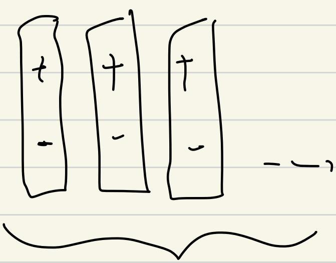
2 physical sources of errors

(1)



switch polarity too often

(2)

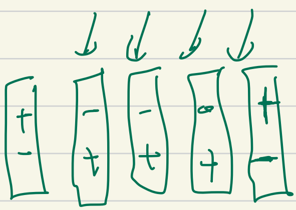


clock that measures when polarity changes clocks drift if you don't change polarity often enough ..

write 1 : switch polarity

" 0 : keep same polarity

1 0 0 1 0 0 0 1 0 0 1



Problem:

how many (2,4) RLL strings of length n are there?

(2,4) - RLL digraph

$$\begin{tikzcd} 1 \arrow[r, "0"] & 2 \arrow[r, "0"] & 3 \arrow[r, "0"] \arrow[ll, bend left, "1"] & 4 \arrow[r, "0"] \arrow[lll, bend left] & 5 \arrow[llll, bend left] \end{tikzcd}$$

$$RLL_{(2,4)}(n) \leq 2^n$$

[In practice

$RLL_{(2,7)}(n)$ worked well]

For any i,

$$RLL_{(2,4)}(n)$$

$$\geq \left(\sum_j\right) \text{Walks}_{i,j} \text{ length } n$$

$$\left(A_{(2,4)}\right)^n_{ij} \leq RLL_{(2,4)}(n)$$

any i,j

$$\leq \sum_{i,j} \left(A_{(2,4)}\right)^n_{ij}$$

$$c_1 \lambda_{PF}^n \leq \left(A_{(2,4)}\right)^n_{ij} \leq c_2 \lambda_{PF}^n$$

λ_{PF} = Perron-Frob eigenvalue of $(2,4)$ graph

$(c_1, c_2 > 0$ constants$)$

$$\lim_{n\to\infty} \left(RLL_{(2,4)}(n) \right)^{1/n} = \lambda_{PF}$$

$$c_1 \lambda_{PF}^n \le RLL_{(2,4)}(n) \le c_2 \cdot 5^2 \cdot \lambda_{PF}^n$$

$$\lambda_{PF}\left(A_{(2,7)}\right) > \sqrt{2}$$

$$\begin{tikzcd}
1 \arrow[r, "0"] & 2 \arrow[r, "0"] & 3 \arrow[r, "0"] \arrow[ll, bend left, "1"] & 4 \arrow[r] \arrow[lll, bend left] & \cdots \arrow[r, "0"] & 8 \arrow[lllll, bend left]
\end{tikzcd}$$

$$\left(A_{(2,7)}^{2n}\right)_{i,j} \geq C \left(\text{larger than } \sqrt{2}\right)^{2n}$$

$$\geq C \left(\text{larger than } 2\right)^{n}$$

So there is a map

$$\underbrace{1011011110\ldots1}_{\substack{\text{strings} \\ \text{length} \\ n}} \longrightarrow \underbrace{}_{\substack{\text{string length} \\ 2n \text{ that} \\ \text{is} \\ (2,7)\text{-RLL}}}$$

Allows encoding

n bits arbitrary (0,1) data $\xrightarrow{\text{encoding}}$ 2n bits of (2,7) RLL (0,1)-data

$\xleftarrow{\text{decoding}}$

(2,7)-RLL run length limited

(2,4)-RLL

(d,k)-RLL

Shannon capacity $\overset{\text{def}}{=} \dfrac{\log \lambda_{PF}(\text{digraph})}{\log 2}$ bits

2^{nd} Application:

Nash Equilibria ...

Idea:

You play the following game

		Reward
You have	option 1	$a_1 \in \mathbb{R}$
	option 2	$a_2 \in \mathbb{R}$
	$\vdots$	$\vdots$
	option n	$a_n \in \mathbb{R}$

option = strategy = choice ...

Reward $\hookrightarrow$ you want to maximize

If $\vec{p} \in \Delta^{n-1} \Longleftrightarrow \vec{p} \in \mathbb{R}^n$,

stochastic

You "randomly" with probability

p_i play option i, worth a_i

$$\text{Reward}(\vec{p}) = \text{Reward}(p_1, \dots, p_n)$$

$$\stackrel{\text{def}}{=} \sum_{i=1}^{n} p_i a_i .$$

$$\mathrm{RewardToSwitch}_{\text{to option } i}(\vec{p})$$

$$= \max\left(0,\ \mathrm{Reward}(\vec{e}_i) - \mathrm{Reward}(\vec{p})\right)$$

$$\vec{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$$

$$\vec{R} \stackrel{\text{def}}{=} (RTS_1, RTS_2, \dots, RTS_n)$$

Claim: Say

$$\vec{p} = \mathrm{Stochastic}(\vec{p} + \varepsilon \vec{R})$$

$\varepsilon > 0$

Then :

① $\vec{R} = c\vec{p}\,, \qquad c \geq 0$

② $c = 0. \quad \vec{R} = 0$

(Also $\vec{p}$: $p_i > 0$

$$a_i = \max_{j=1,\dots,n} (a_j)$$