CPSC 531F March 21, 2025

- Reward to Switch + Nash Equilibria

- Δ-complexes $\longrightarrow$ Point Clouds

generalization of simplicial complexes, need to make examples

feasible to work with

Last time:

1-player, n-strategy game:

given $a_1, \ldots, a_n \in \mathbb{R}$

mixed strategy: $\vec{p} \in \mathbb{R}^n$, stochastic,

$(\vec{p} \in \Delta^{n-1})$

$$\text{Reward}(\vec{p}) = p_1 a_1 + p_2 a_2 + \cdots + p_n a_n$$

Highest possible reward:

$$a_1 \geq a_2 \geq \cdots \geq a_n$$

Play: $\vec{e}_1 = (1, 0, \ldots, 0)$, Reward $= a_1$.

Say... $a_1 = 2, \quad a_2 = 1, \quad a_3 = 0$

$\text{Reward}(p_1, p_2, p_3)$

$$= 2p_1 + 1 \cdot p_2 + 0 \cdot p_3$$

Define:

$$\overrightarrow{\text{RewardToSwitch}}(\vec{p})$$

$$= \left(\text{RewToSw}_{\text{to } 1}, \text{RTS}_{\text{to } 2}, \text{RTS}_{\text{to } 3} \right)$$

Nash!

$$\text{RewToSw}_{\text{to } i} \overset{\text{Nash}}{=} \quad \text{4,} \quad a_i$$

$$\max\left(0, \text{Reward}(\vec{e_i}) - \text{Reward}(\vec{p}) \right)$$

Example:

$$p_1 = \frac{1}{5}, \quad p_2 = \frac{1}{5}, \quad p_3 = \frac{3}{5}$$

$$\text{Reward}(p_1, p_2, p_3)$$

$$= \text{Reward}\left(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}\right):$$

$$\text{Reward}\left(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}\right) = 2 \cdot \frac{1}{5} + 1 \cdot \frac{1}{5} = \frac{3}{5}$$

$$0 = a_3 < \frac{3}{5} < a_2 = 1 < a_1 = 2$$

$$\text{RewardToSw}: \left(2 - \frac{3}{5},\ 1 - \frac{3}{5},\ 0\right)$$

For any $\vec{p}$

1. $\mathrm{RewToSw}(\vec{p}) \in \mathbb{R}^3_{\geq 0}$

non-neg comp

2. $\forall i \in [n]$

$a_i = \mathrm{Reward}(\vec{e}_i) \leq \mathrm{Reward}(\vec{p})$

$\Longrightarrow$

$\mathrm{RewToSw}_{\text{to } i} = 0$

3. $\forall i \in [n]$

$a_i = \mathrm{Reward}(\vec{e}_i) > \mathrm{Reward}(\vec{p})$

$\mathrm{RewToSw}_{\text{to } i} > 0.$

4. RewToSw is continuous

Idea: Start at any $\vec{p}^0 \in \Delta^{n-1}$

$$\vec{p}^1 = \text{Stoch}\left(\vec{p}^0 + \text{RewToSw}(\vec{p}^0)\right)$$

$$\vec{p}^2 = \text{---} \quad \vec{p}^1 \qquad\qquad \vec{p}^1 \;)$$

$\vdots$

want to converge ...

<u>Remark</u>:

$$\text{RewToSwitch} = \vec{0}$$

$$\Rightarrow \text{Reward}(\vec{p}) \text{ is } \begin{array}{l} \geq a_1 \\ \geq a_2 \\ \vdots \\ \geq a_n \end{array}$$

$a_1 = a_2 = a_3 = 10, \quad a_4 = 1, \quad a_5 = 0$

$$\text{Reward}(\vec{p}) \geq \max(a_1, \ldots, a_n)$$

$\Leftrightarrow$

when $p_i > 0$, $\quad a_i = \max\limits_{j=1,\ldots,n} a_j$

<u>Remark 2:</u>

If $\text{RewToSwitch}(\vec{p}) \neq 0$ then

$$\text{Reward}\left(\text{Stoch}\left(\text{RewToSw}(\vec{p})\right)\right) > \text{Reward}(\vec{p}))$$

For > 2025:

Is Nash's RewToSw convex?

Assuming

cont

continuous

$$\vec{p}^{\,m+1} = \mathrm{Stoch}\left(\vec{p}^{\,m} + \mathrm{RewToSw}(\vec{p}^{\,m})\right)$$

limit

$$\vec{p}^{\,m} \xrightarrow{m\to\infty} \vec{p}$$

$$\vec{p} \qquad \vec{p} \qquad \vec{p}$$

$$\boxed{\vec{p} = \mathrm{Stoch}\left(\vec{p} + \mathrm{RewToSw}(\vec{p})\right)}$$

Maybe $\vec{p}^{n} \xrightarrow{n\to\infty}$ has a limit
maybe no

-- —

$\boxed{\text{Brouwer fixed point thm:}}$

$$f(\vec{p}) = \mathrm{Stoch}\left(\vec{p} + \mathrm{RewToSw}(\vec{p})\right)$$

$$f : \Delta^{n-1} = \{\text{Stochastic vectors in } \mathbb{R}^n\} \longrightarrow \Delta^{n-1}$$

And so $f(\vec{p}) = \vec{p}$ for some $\vec{p}$.

Hence ...

For some $\vec{p} \in \Delta^{n-1}$,

$$\vec{p} = f(\vec{p})$$

$$= \text{Stoch}\left(\vec{p} + \text{RowToSw}(\vec{p})\right)$$

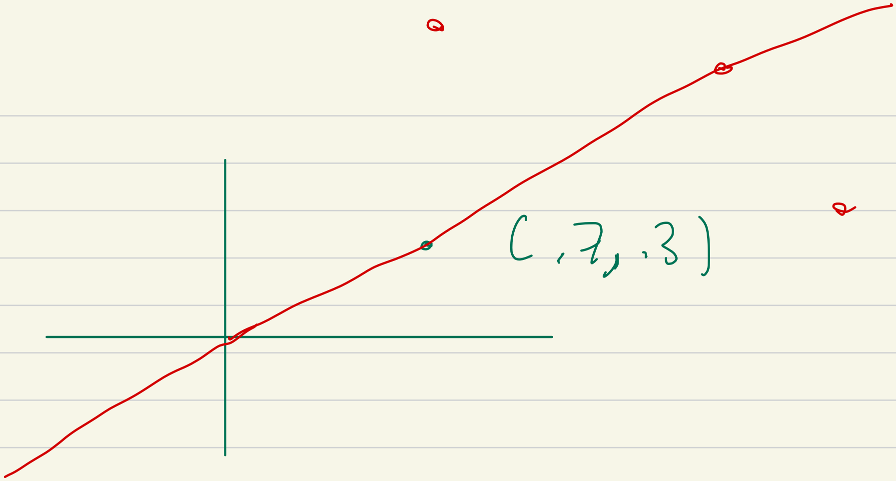
So:

$$\vec{p} = \text{Stoch}\left(\vec{p} + \vec{R}\right)$$

($n=2$, example

$$(.7, .3) = \text{Stoch}\left((.7, .3) + \vec{R}\right)$$

non-neg comp



$\Rightarrow \quad \vec{R} = c\,\vec{p}$

So, for n-strategies:

$\overrightarrow{\text{RewardToSwitch}} = c\,\vec{p}$

Say $a_1 \geq a_2 \geq \dots \geq a_n$

Say $\underbrace{a_1 = a_2 = \dots = a_r}_{\max} > a_{r+1} \geq \dots$

Then! Claim: $c = 0$ in (2)

$$\overrightarrow{\text{RewardToSwitch}} = c(p_1, \dots, p_r, \underbrace{p_{r+1}, \dots, p_n}_{\text{these have to be zero}})$$

if this is > 0

Then, say that j is the maximum value where $p_j > 0$

$$a_1 = a_2 = \dots = a_r > a_{r+1} \geq a_{r+2} \geq \dots \geq a_j$$

$$p_1 \quad p_2 \quad \dots \quad p_{j+1} \qquad p_j > 0$$

$p_{j+1}, \dots, p_n = 0$

So ... whenever

$$\vec{p} = \mathrm{Stoch}\Big(\vec{p} + \underbrace{\mathrm{RewToSw}(\vec{p})}\Big)$$

(1) $= c\,\vec{p}$

(2) $c = 0$

$\Rightarrow$ $\mathrm{RewToSw}(\vec{p}) = (0, 0, \ldots, 0)$

$\Rightarrow$ $\mathrm{Reward}(\vec{p}) = \max\limits_{j=1,\ldots,n} a_j$

$\Rightarrow$ at your 1-player "Nash equilibrium"

Claim! Same idea

$\Longrightarrow$ any k-player game

there is some mixed strategy

of each of the k-players

where each player is

at an equilibrium --

	R	P	S
R	(0,0)	.	)
P	(1,-1)	(0,0)	`
S	-	-	`

Drive	R	L
R	(0,0)	(-1,-1)
L	(-1,-1)	(0,0)

1 - Nash eq

2 - Nash equib

$(\vec{e_1}, \vec{e_1})$

$(\vec{e_2}, \vec{e_2})$

(0, 0) (-1, -1)

(-1, -1) (.01, .01)