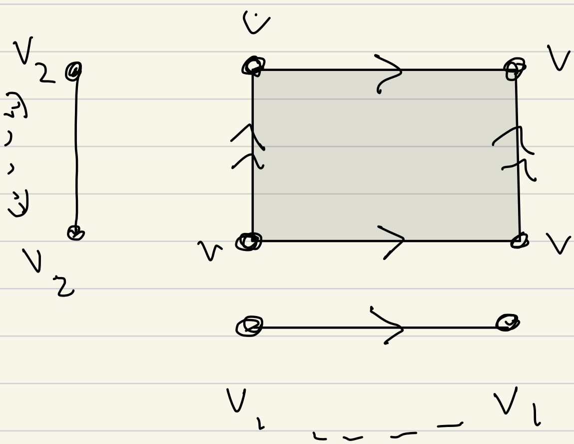
CPSC 531F March 24, 2025

- Δ-complexes
- Homology Barcodes

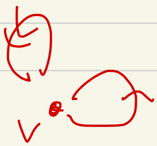
Torus: $S^1 \times S^1 = T$



$v = (v_1, v_2)$

can't be made into a simplicial complex

can have



Thm:

$$\beta_i(X \times Y) = \sum_{i_1+i_2=i} \beta_{i_1}(X)\,\beta_{i_2}(Y)$$

example

$$\beta_0(S^1) = 1, \quad \beta_1(S^1) = 1$$

$$\beta_{\cancel{2}} = \beta_3 = \cdots = 0$$

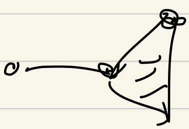
$$\beta_0(T) = \beta_0(S^1)\,\beta_0(S^1) = 1$$

$$\beta_1(T) = \beta_0(S^1)\,\beta_1(S^1) + \beta_1(S^1)\,\beta_0(S^1) = 1+1=2$$

$$\beta_2(T) = \beta_1(S^1)\,\beta_1(S^1) = 1$$

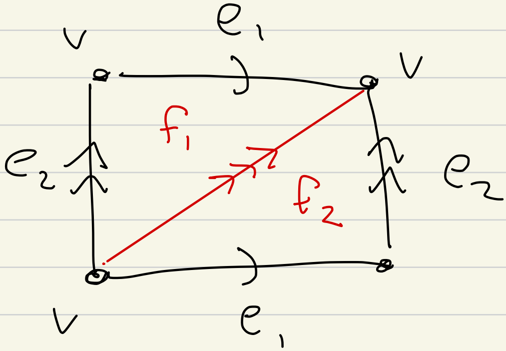
If K is a simplicial complex in $\mathbb{R}^N$, then

$$H_i\left(|K|_{geom}\right) \overset{\text{THM}}{=} H_i\left(K^{abs}\right)$$



$\subset \mathbb{R}^N$

Idea:



Torus, in this way, becomes

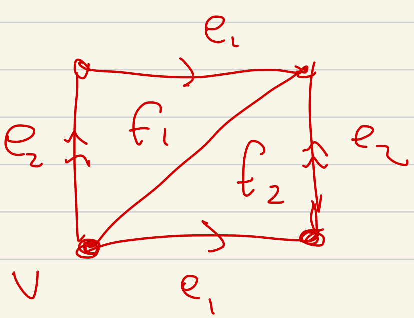
$\sigma_v : \Delta^0 \longrightarrow T$

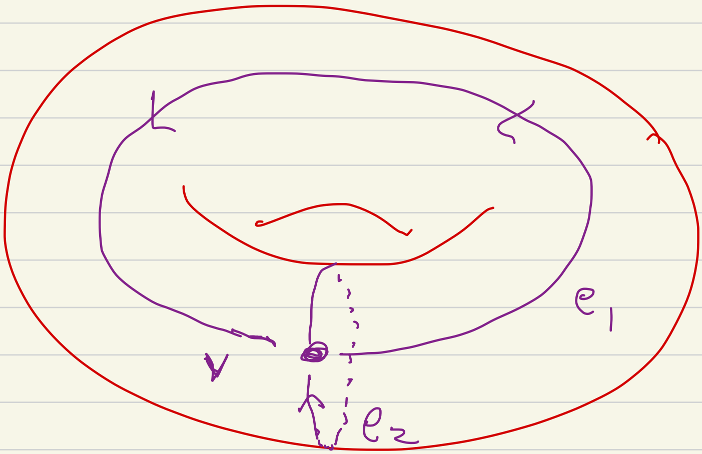
$\rotatebox{90}{$\in$}$

a point $\longmapsto v$

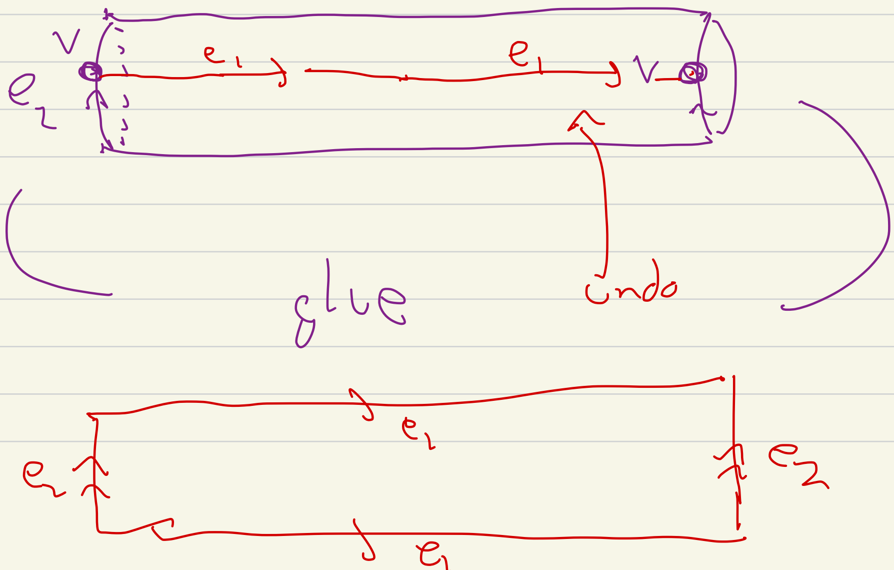
$\sigma_{e_1} : \Delta^1 \longrightarrow T$

$\sigma_{e_2} : \Delta^1 \longrightarrow T$



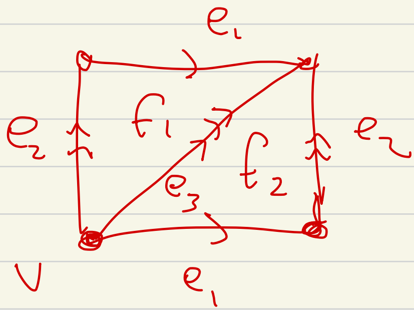


undo e_2



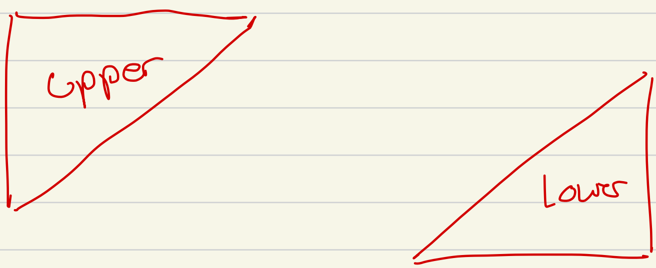
$\sigma_v : \Delta^0 \longrightarrow T$

$\sigma_{e_1}, \sigma_{e_2}, \sigma_{e_3} : \Delta^1 \longrightarrow T$



$f_1, f_2 : \Delta^2 \longrightarrow T$

f_1



f_2

① $\sigma_\alpha : \Delta^{n_\alpha} \longrightarrow T$ s.t.

$\sigma_\alpha : (\Delta^{n_\alpha})^{\text{interior}} \xrightarrow{\text{injection}} T$

② Every point of T is

either:

- a vertex $\left(\sigma_{v_i} : \Delta^0 \to T\right)$
- in the interior of

$$\sigma_\alpha : (\Delta^{n_\alpha})^{\text{interior}} \longrightarrow T$$

$n_\alpha \geq 1$

③ $\sigma_\alpha : \Delta^{n_\alpha} \to T$

look at each boundary element

is one of the simplexes.

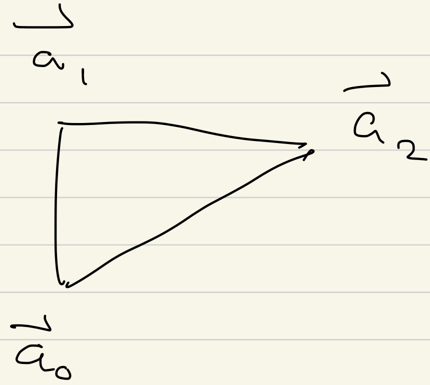
④ $U \subset T$ open iff

$U \cap \mathrm{Image}(\sigma_\alpha)$ is open,

Δ-complex section in this handout, reference: textbook by Hatcher.

Point ③ is a bit subtle...

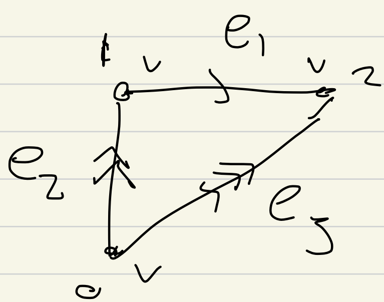
f_1 :

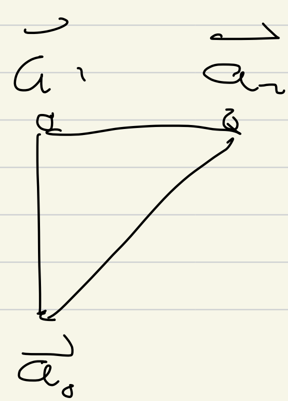


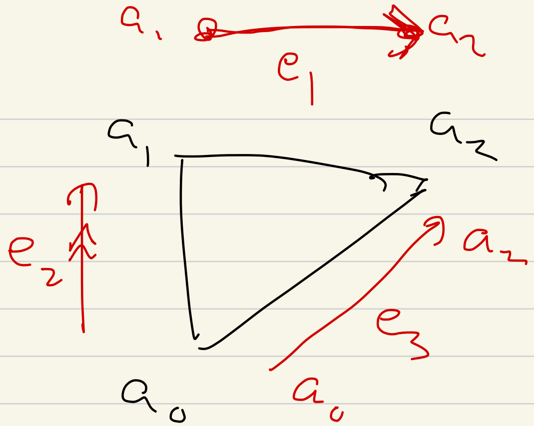
(Rem: We'll use $\vec{a}_0, \vec{a}_1, \vec{a}_2$, since $\vec{e}_1, \vec{e}_2, \vec{e}_3$ for Δ^2 we might confuse $\vec{e}_i$ with edges)

Other textbooks

f_1







boundary :

$\cancel{a_0}\ a_1\ a_2$

$a_0\ \cancel{a_1}\ a_2$

$a_0\ a_1\ \cancel{a_2}$

$a_1 a_2$ boundary

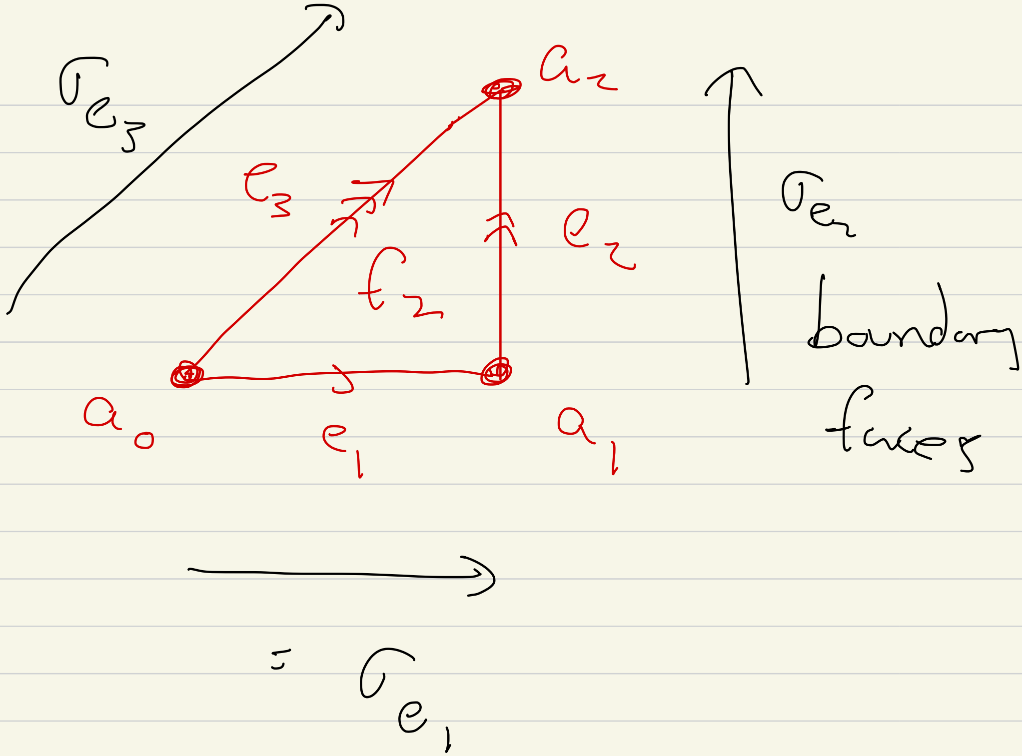
map,

restriction of σ_{f_1}

$= \sigma_{e_1} : \Delta^1 \longrightarrow T$

Restrict σ_{f_1} to $a_0, a_2 = \Delta_{e_2}$

" " to $a_0, a_2 = \Delta_{e_3}$



a_2
σ_{e_3}
e_3
e_2
f_2
σ_{e_2}
boundary faces
a_0
e_1
a_1
$= \sigma_{e_1}$

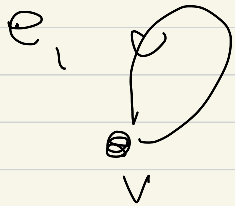
Define the "simplicial homology of Δ-complex structure on T!

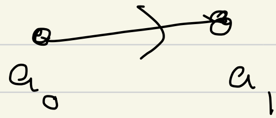
2-dim maps $\sigma_{f_1}, \sigma_{f_2}$ 2-simplex

1-dim maps $\sigma_{e_1}, \sigma_{e_2}, \sigma_{e_3}$ 1-simplex

0-dim maps σ_{v_0} 0-simplex

But boundary maps





$\partial(\sigma_{e_1}) =$ map's restriction to a_1 $-$ map's restriction to a_0

$\sigma_{v_0} - \sigma_{v_0}$

$C_2(\Delta\text{-complex structure}) = \mathbb{R}$-linear combos of $\sigma_{f_1}, \sigma_{f_2}$

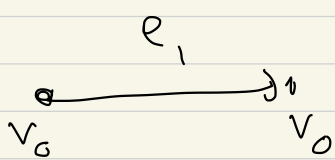
$C_1(\qquad) = \mathbb{R}$-linear combos of $\sigma_{e_1}, \sigma_{e_2}, \sigma_{e_3}$

$$\mathcal{C}_0(\quad) = \mathbb{R}\text{-linear combo's of } \sigma_{v_0}$$

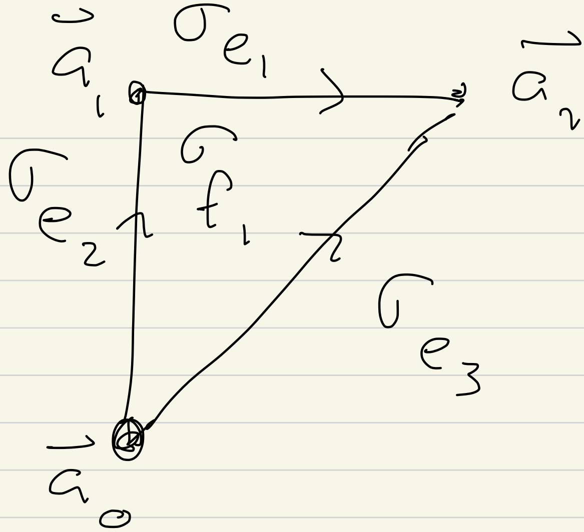
$$\mathcal{C}_2 \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \xrightarrow{\partial_0} 0$$

$\mathbb{R}\sigma_{f_1} + \mathbb{R}\sigma_{f_2}$	$\mathbb{R}\sigma_{e_1} + \mathbb{R}\sigma_{e_2} + \mathbb{R}\sigma_{e_3}$	$\mathbb{R}\sigma_{v_0}$

$$\partial_1(\sigma_{e_1}) = \sigma_{v_0} - \sigma_{v_0} = 0$$



$$\partial(\sigma_{e_2}) = \partial(\sigma_{e_3}) = 0$$

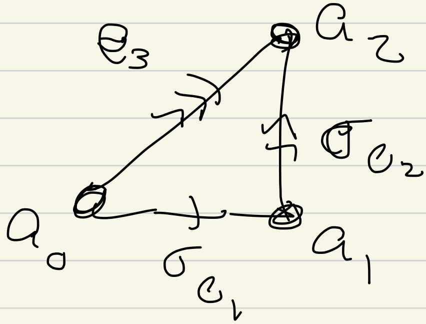


$\cancel{a_0}\, a_1\, a_2$

$a_0\, \cancel{a_1}\, a_2$

$a_0\, a_1\, \cancel{a_2}$

$$\partial \sigma_{f_1} = \sigma_{e_1} - \sigma_{e_3} + \sigma_{e_2}$$



$$\partial \sigma_{f_2} = \sigma_{e_2} - \sigma_{e_3} + \sigma_{e_1}$$

$\mathcal{C}_2 \qquad \mathcal{C}_1 \qquad \mathcal{C}_0$

$\mathbb{R}^2 \qquad \mathbb{R}^3 \qquad \mathbb{R}$

$1 \to \sigma_{e_1} \mapsto 0$

$\sigma_{E_1} \longrightarrow \quad 1 \longrightarrow \sigma_{e_2} \mapsto 0$

$\sigma_{E_2} \longrightarrow \quad (-1) \longrightarrow \sigma_{e_3} \mapsto 0$

So

$$\mathbb{R}^2 \xrightarrow{\partial_2} \mathbb{R}^3$$

image ∂_2 dim 1

ker ∂_2 dim 1

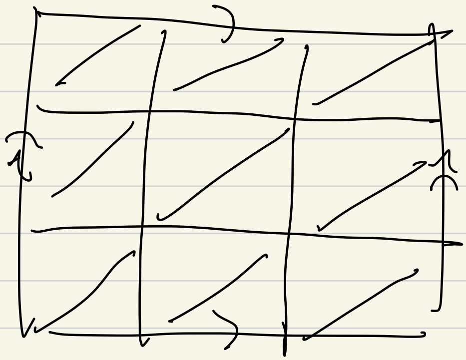
so get

$\beta_0 = 1$

$\beta_1 = 2$

$\beta_2 = 1$

Simplicial complex



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2-simplices