CPSC 531F March 26, 2025

- Persistent Topology:

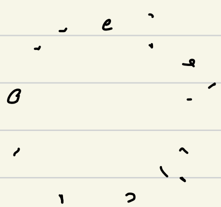
  - As defined in Edelsbrunner, Letscher, Zomorodian, 2002, Topological Persistence & Simplification (top page 516 there)

  - Simple examples

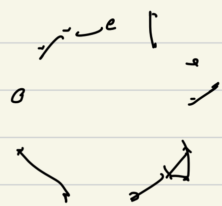
  - Barcodes of the examples

  - General theory of barcodes

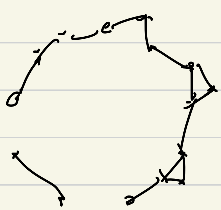
Idea! You have a point cloud, finite set, V, in $\mathbb{R}^N$



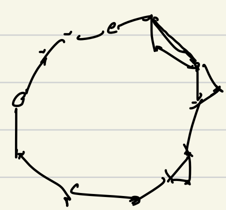
$V = K^0_{abs}$



K^1_{abs}



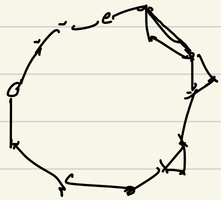
K^2_{abs}



K^3_{abs}

$H(K^2_{abs}) \simeq \mathbb{R}^2$

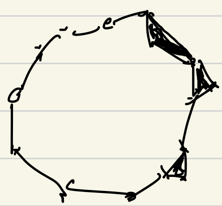
$H_1(K^3_{abs}) \simeq \mathbb{R}^4$



K^3_{abs} as a graph

$H_1(K^3_{abs}) \simeq \mathbb{R}^4$

Vietoris-Rips Complex associated to

$H_1\Big($  $\Big) \simeq \mathbb{R}$

We hope this cycle "persists" as we add more edges to graph

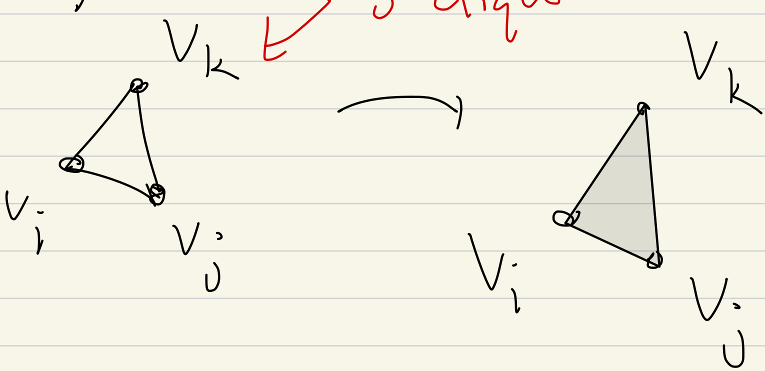
Idea: If $G=(V,E)$ is a graph, then

↑ simple

Vietoris-Rips Complex of G:

$$\{ C \subset \bar{V} \mid C \text{ is a clique in } G \}$$

e.g.



C is a clique in $G=(\bar{V},E)$

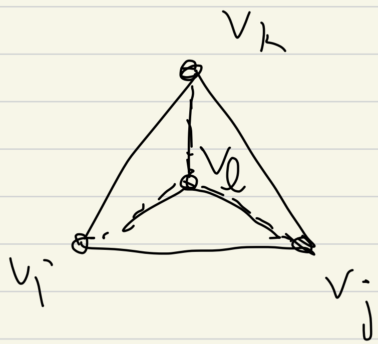
if $C \subset V$ and

$\forall\, c_1, c_2 \in C$, $c_1 \neq c_2$,

there's an edge with endpoints

c_1 and c_2

4-clique



$\subset \mathbb{R}^3$

↳ add $\{v_i, v_j, v_k, v_\ell\}$

to Vietoris-Rips

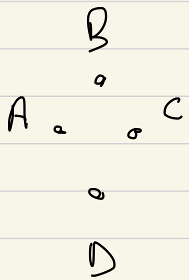
Complex of G

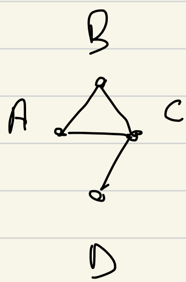
Question! Say we have simplicial complexes

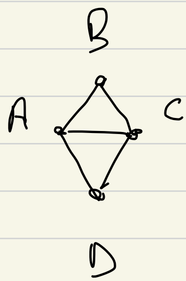
$$K^0_{abs} \subset K^1_{abs} \subset \cdots$$

What does it mean to speak of $H_1(K^i_{abs})$ elements that "persist"?

Question 1: Consider:

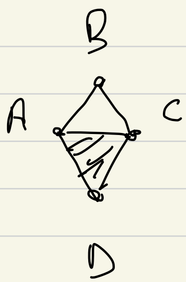


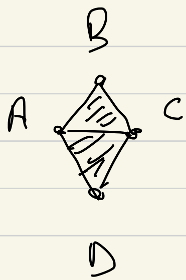




$K^0_{abs} \subset K^1_{abs} \subset K^2_{abs} \subset$

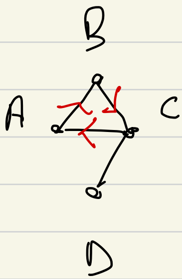
$\beta_0 = 4$, $\beta_1 = 0$ $\beta_0 = 1$, $\beta_1 = 1$ $\beta_1 = 2$

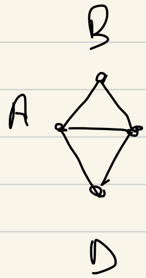


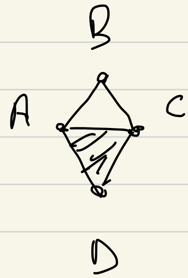


$\subset K^3_{abs} \subset K^4_{abs}$

$\beta_1 = 1$ $\beta_1 = 0$







$$K^1_{abs} \subset K^2_{abs} \subset K^3_{abs}$$

$\beta_0 = 1$, $\beta_1 = 1$ $\qquad$ $\beta_1 = 2$ $\qquad$ $\beta_1 = 1$

$$H_1(K^1_{abs}) = \mathbb{R}\,\tau$$

$\tau = [A,C] + [C,D] + [D,A] \mapsto 0$

$\tau = [A,B] + [B,C] + [C,A]$

$\tau = [A,B] + [B,C] + [C,A] \mapsto \tau \neq 0$ in $H_1(K^3_{abs})$

$\partial_1 \tau = 0$

Is there a persistent $H_1(K^i)$ element?

Yes;

$$\tau = [A,B] + [B,C] + [C,A]$$

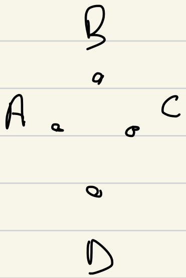
$\partial_1 \tau = 0$, τ lives in K^i_{abs}

and $\tau \neq 0$ in $H_1(K^1_{abs})$

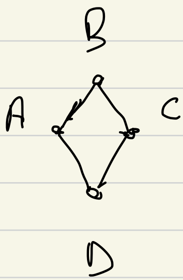
$H_1(K^2_{abs})$

$H_1(K^3_{abs})$

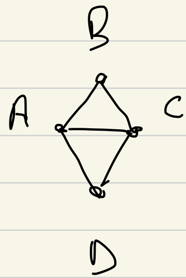
Question 2: Consider:



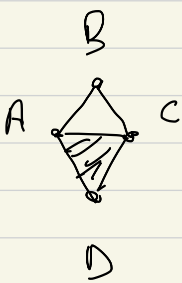
K^0_{abs}



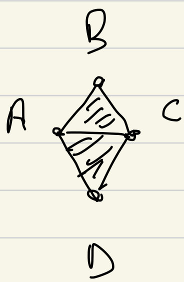
K^1_{abs}



K^2_{abs}



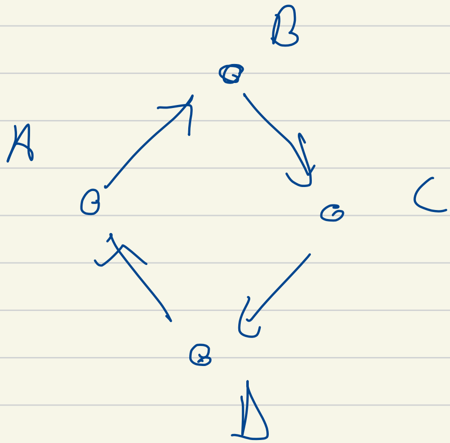
K^3_{abs}



K^4_{abs}

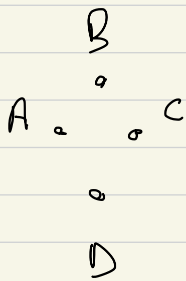
Is some $H_1(K^i_{abs})$ element the "most persistent"?

In K^1_{abs}

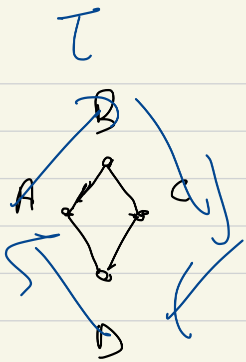


$\tau = [A,B] + [B,C] + [C,D] + [D,A]$

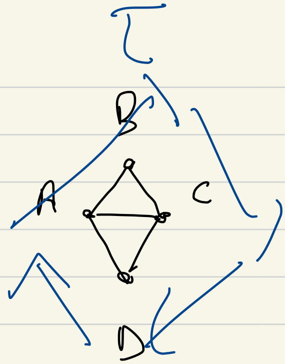
$\partial_1 \tau = 0$, $\tau \neq 0$ in $H_1(K^1_{abs})$



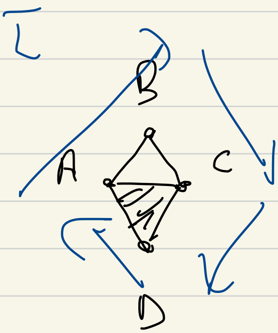
K^0_{abs}



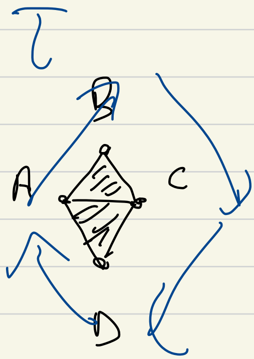
K^1_{abs}



K^2_{abs}

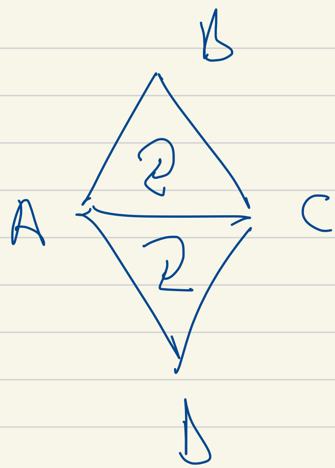


K^3_{abs}



K^4_{abs}

In K^4_{abs}, that $\tau = \partial_2\left([A,B,C] + [A,C,D] \right)$

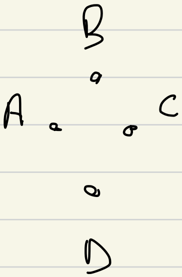


$$\partial_2\left([A,B,C] + [A,C,D] \right) = [A,B] + [B,C] + \cancel{[C,A]} + \cancel{[A,C]} + [C,D] + [D,A]$$

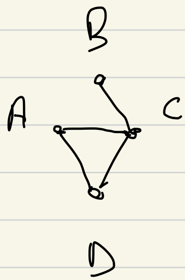
$$= \text{in } [A,B] + [B,C] + [C,D] + [D,A] \quad \text{in } C_1$$

$$= \tau$$

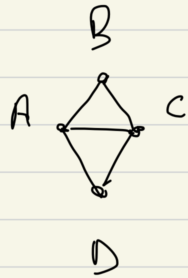
Question 3: Consider!



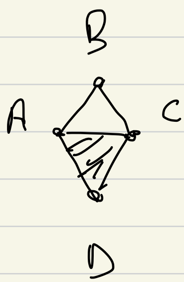
K^0_{abs}



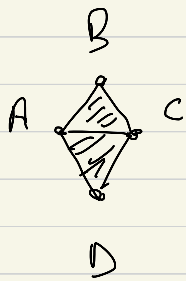
K^1_{abs}



K^2_{abs}

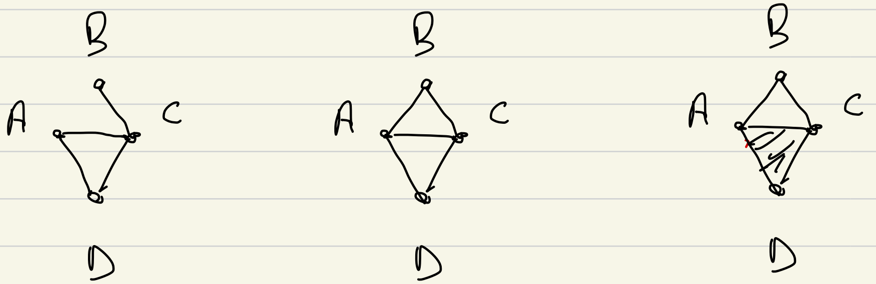


K^3_{abs}



K^4_{abs}

Is there a most persistent element of $H_1(K^i_{abs})$?



K^1_{abs} K^2_{abs} K^3_{abs}

some cycles are new here $\longrightarrow$ stay nonzero in H_1

$\left.\begin{array}{l}[A,C]+\\ [C,D]+\\ [D,A]\end{array}\right\}\tau \in H_1(K^1)$ $\longrightarrow$ $\tau \neq 0$ in $H_1(K^2)$ $\longrightarrow$ $\tau = 0$ in $H_1(K^3)$

$\tau = \partial_2 [A,C,D]$

Idea! "Barcode":

Say we have a set of vector spaces $V^0, V^1, \ldots, V^n$

(e.g. $H_1(K^0_{abr}), H_1(K^1_{abr}), \ldots$)

Say also we have linear maps

$$V^0 \xrightarrow{L^0} V^1 \xrightarrow{L^1} V^2 \to \ldots \xrightarrow{L^{n-1}} V_n$$

Say: if $v_0 \in V^0$, and

$L^0 v_0 \neq 0$ in V^1, $L^1 L^0 v_0 \neq 0$ in V^2

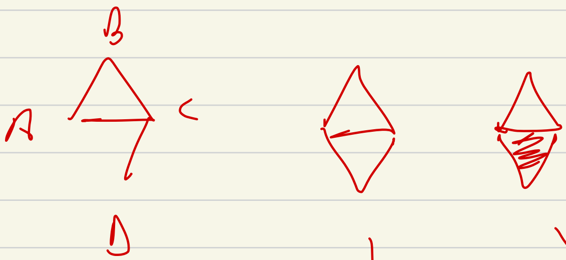
...

and $\mathcal{L}_{n-1} \cdots R_1 R_0 v_0 \in \bar{V}^n$

is non-zero,

we say v_0 "fully persistr"

Example



$H_1(\ )$ $\quad$ $H_1(\ )$ $\quad$ $H_1(\ )$

$$V^0 \xrightarrow{\mathcal{L}^0} V^1 \xrightarrow{\mathcal{L}^1} V^2$$

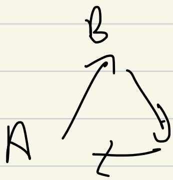
dim = 1 $\quad$ dim 2 $\quad$ dim 1

persists

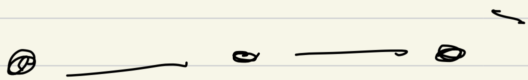
$\tau = (A,B)$
$+ (B,C)$
$+ (C,A)$

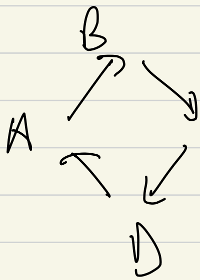
$\mathcal{L}^0(\tau)$ $\quad$ $\mathcal{L}^1\mathcal{L}^0(\tau)$

Describe: $V^0 \xrightarrow{\mathcal{L}^0} V^1 \xrightarrow{\mathcal{L}^1} V^2$

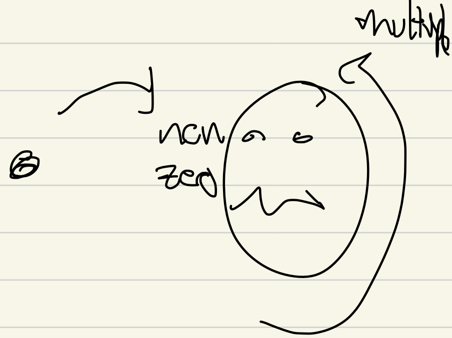


$= \tau$ in



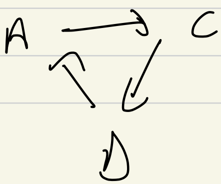


$= \tau$



$$\mathcal{L}^1\mathcal{L}^0(\tau) - \mathcal{L}^1\mathcal{L}^0(\tau_{☺}) = 0$$

in V^2



doesn't exist

$\bullet \longrightarrow$ zero